

1086



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1205

EXACT CALCULATION OF LAMINAR BOUNDARY LAYER IN
LONGITUDINAL FLOW OVER A FLAT PLATE
WITH HOMOGENEOUS SUCTION

By Rudolf Iglisch

Translation of "Exakte Berechnung der laminaren Grenzschicht an der
längsangeströmten ebenen Platte mit homogener Absaugung."
Schriften der Deutschen Akademie der Luftfahrtforschung,
Band 8 B, Heft 1, 1944.



Washington

April 1949

AFMDC
TECHNICAL LIBRARY
AFL 2311



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 1205

EXACT CALCULATION OF LAMINAR BOUNDARY LAYER IN
LONGITUDINAL FLOW OVER A FLAT PLATE
WITH HOMOGENEOUS SUCTION*

By Rudolf Iglisch

1. Statement of the Problem and Introduction

Lately it has been proposed (reference 1) to reduce the friction drag of a body in a flow for the technically important large Reynolds numbers by the following expedient: the boundary layer, normally turbulent, is artificially kept laminar up to high Reynolds numbers by suction. The reduction in friction drag thus obtained is of the order of magnitude of 60 to 80 percent of the turbulent friction drag, since the latter, for large Reynolds numbers, is several times the laminar friction drag. In considering the idea mentioned one has first to consider whether suction is a possible means of keeping the boundary layer laminar. This question can be answered by a theoretical investigation of the stability of the laminar boundary layer with suction. A knowledge, as accurate as possible, of the velocity distribution in the laminar boundary layer with suction forms the starting point for the stability investigation. H. Schlichting (reference 2) recently gave a survey of the present state of calculation of the laminar boundary layer with suction.

The classic example for Prandtl's theory of the boundary layer without suction (reference 3) is the boundary layer, calculated by H. Blasius (reference 4), which develops on an infinitely extended flat plate in longitudinal flow. Again and again one reverts for comparison to this example, even where the lay-out of the problems is entirely different. Study of the flow along the flat plate will thus have the same importance and significance for the boundary layer with suction or blowing. The developing boundary layer will, of course, no longer depend only on the shape of the body immersed in the flow - a plate in the present study - but also on the manner of suction or blowing. The case of

*"Exakte Berechnung der laminaren Grenzschicht an der längs-angeströmten ebenen Platte mit homogener Absaugung." Schriften der Deutschen Akademie der Luftfahrtforschung, Band 8 B, Heft 1, 1944. Report of the Mathematical Institute of the Technical Academy Braunschweig. Presented to the German Academy of Aviation Research by the corresponding member Hermann Schlichting on January 26, 1944.

continuous suction with constant suction velocity will be of particular theoretical importance; it is numerically calculated in detail in the present report and for the general case of arbitrary suction or blowing the complete system of formulas is given with which the numerical calculation may be performed in exactly the same manner.

The plate is assumed to extend an infinite distance in the x direction, starting at the origin of coordinates (fig. 1). For negative x the flow is then undisturbed; thus $u = U_0$, $v = 0$, where u and v signify the velocity components in the x and y direction, respectively, and U_0 is the free stream velocity. The equation

$$v(x,0) = v_0(x) \quad (1)$$

gives the prescribed suction law; $v_0(x) < 0$ signifies suction, $v_0(x) > 0$ blowing. The condition $u(x,0) = 0$ at the wall is also to be maintained for the boundary layer with suction or blowing. Prandtl's boundary layer equations (references 4 and 5) (with the sink effect neglected) read

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

ν signifies the kinematic viscosity. The boundary conditions (fig. 2)

$$u(x,0) = 0, \quad v(x,0) = v_0(x), \quad u(x,\infty) = U_0, \quad u(0,y) = U_0 \quad (4)$$

also have to be satisfied. A special very simple solution of the system of equations (2) to (4) with the exception of the fourth boundary condition (4) can be given for the case of continuous homogeneous suction; that is for $v_0(x) = v_0 = \text{const.} < 0$. In this case, according to H. Schlichting, (reference 6)

$$u(x,y) = U_0 \left(1 - e^{-\frac{v_0 y}{\nu}} \right), \quad v(x,y) = v_0 \quad (5)$$

Thus, velocity distribution and hence also boundary layer thickness and all remaining boundary layer parameters are independent of the distance along the plate x for this solution; it is called the asymptotic solution.

For a plate provided with homogeneous suction beginning at the leading edge the boundary layer which starts in front with zero thickness must be transformed, after a certain distance, into this asymptotic solution. This distance in particular will be treated in detail below.

In the first chapter (paragraphs 2 to 5) the general equations for the solution of the complete problem (equations (2), (3), and (4)) are set up; in chapter two (paragraphs 6 to 18) the case $v_0(x) = v_0$ with $v_0 < 0$, that is, the case of constant suction, is treated numerically. The calculation methods described here can, of course, be applied also in the case of a general suction law. Furthermore, the validity of these considerations is, naturally, not limited to the flow over a plate¹; the numerical methods described may be used also for airfoil profiles.

I. THE GENERAL SYSTEM OF FORMULAS

2. Elimination of the Kinematic Viscosity ν and the Free Stream Velocity U_0

The equations (2) and (3) are simplified, if one inserts

$$x = x_1 \nu; \quad y = y_1 \nu \quad (6)$$

with

$$u_1(x_1, y_1) = u(x, y) \quad \text{and} \quad v_1(x_1, y_1) = v(x, y)$$

to

¹This investigation has been suggested to me by Mr. H. Schlichting.

$$\left. \begin{aligned} u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} &= \frac{\partial^2 u_1}{\partial y_1^2} \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} &= 0 \end{aligned} \right\} \quad (7)$$

according to equation (4) the boundary conditions

$$\left. \begin{aligned} u_1(x_1, 0) &= 0, \quad v_1(x_1, 0) = v_0(x_1, v) = v_{01}(x_1), \\ u_1(x_1, \infty) &= U_0, \quad u_1(0, y_1) = U_0 \end{aligned} \right\} \quad (8)$$

also are to be satisfied. If one makes the new substitution

$$\left. \begin{aligned} x_2 &= \frac{2}{U_0} x_1, \quad y_2 = y_1, \\ u_2(x_2, y_2) &= \frac{2}{U_0} u_1(x_1, y_1), \quad v_2(x_2, y_2) = v_1(x_1, y_1), \end{aligned} \right\} \quad (9)$$

equation (7) is transformed into

$$u_2 \frac{\partial u_2}{\partial x_2} + v_2 \frac{\partial u_2}{\partial y_2} = \frac{\partial^2 u_2}{\partial y_2^2} \quad (10)$$

$$\frac{\partial u_2}{\partial x_2} + \frac{\partial v_2}{\partial y_2} = 0 \quad (11)$$

with the boundary conditions according to (8)

$$\left. \begin{aligned} u_2(x_2, 0) = 0, v_2(x_2, 0) &= v_{o1} \left(\frac{U_o}{2} x_2 \right) = v_{o2}(x_2), \\ u_2(x_2, \infty) = 2, u_2(0, y_2) &= 2 \end{aligned} \right\} \quad (12)$$

3. Introduction of the Stream Function ψ_2 as
Independent Variable in Addition to x_2

If one puts

$$u_2 = \frac{\partial \psi_2}{\partial y_2}, v_2 = -\frac{\partial \psi_2}{\partial x_2} \quad (13)$$

equation (11) is automatically satisfied. With

$$u_2(x_2, y_2) = U_2(\psi_2, x_2) \quad (14)$$

equation (10) is transformed into

$$\frac{\partial U_2}{\partial x_2} = \frac{\partial^2 U_2}{\partial \psi_2^2} U_2 + \left(\frac{\partial U_2}{\partial \psi_2} \right)^2$$

which is equivalent to the equation

$$\frac{\partial z}{\partial x_2} = \sqrt{z} \frac{\partial^2 z}{\partial \psi_2^2} \quad (15)$$

(see reference 7, p. 847, formula 9.) if one puts

$$z(\psi_2, x_2) = U_2^2(\psi_2, x_2) \quad (16)$$

Because of equations (12), (13), and (16), the following boundary conditions (cf. fig. 3 where a sink distribution is assumed) must be satisfied, aside from (15):

$$z(\infty, x_2) = 4 = z(\psi_2, 0), \quad z[\bar{f}(x_2), x_2] = 0 \quad (17)$$

with

$$f(x_2) = -\int_0^{x_2} v_{o2}(x_2) dx_2 \quad (18)$$

The reconversion from the coordinates ψ_2, x_2 to x_2, y_2 is made according to (13) by

$$y_2 = \int_{f(x_2)}^{\psi_2} \frac{d\psi_2}{U_2(\psi_2, x_2)} = \int_{f(x_2)}^{\psi_2} \frac{d\psi_2}{\sqrt{z(\psi_2, x_2)}} \quad (19)$$

(for constant x_2).

4. Simplification of the Limits of the Region of Integration

The region of integration obtains straight-line boundaries only, if one introduces instead of x_2 and ψ_2 the new variables

$$\xi_1 = \psi_2 - f(x_2), \quad \eta_1 = \int_0^{x_2} F(x_2) dx_2 \quad (20)$$

with

$$F(x_2) = \sqrt{1 + f'^2(x_2)} \quad (21)$$

(cf. fig. 4). One may write the second equation (20) abbreviatedly in the form

$$\eta_1 = s(x_2) \text{ with the solution } x_2 = S(\eta_1) \quad (22)$$

With

$$z(\psi_2, x_2) = Z(\xi_1, \eta_1) \quad (23)$$

because of

$$\frac{\partial z}{\partial x_2} = -\frac{\partial Z}{\partial \xi_1} f'(x_2) + \frac{\partial Z}{\partial \eta_1} F(x_2), \quad \frac{\partial z}{\partial \psi_2} = \frac{\partial Z}{\partial \xi_1}$$

equation (15) becomes

$$-\frac{\partial Z}{\partial \xi_1} f'(x_2) + \frac{\partial Z}{\partial \eta_1} F(x_2) = \sqrt{Z} \frac{\partial^2 Z}{\partial \xi_1^2} \quad (24)$$

If one designates, according to (22),

$$-f'(x_2) = -f'[S(\eta_1)] = a(\eta_1), \quad (25)$$

$$F(x_2) = F[S(\eta_1)] = b(\eta_1), \quad (26)$$

equation (24) becomes

$$a(\eta_1) \frac{\partial Z}{\partial \xi_1} + b(\eta_1) \frac{\partial Z}{\partial \eta_1} = \sqrt{Z} \frac{\partial^2 Z}{\partial \xi_1^2} \quad (27)$$

Furthermore, according to (17) the boundary conditions (cf. fig. 5)

$$Z(\infty, \eta_1) = 4 = Z(\xi_1, 0), \quad Z(0, \eta_1) = 0 \quad (28)$$

must be satisfied.

5. Elimination of the Discontinuity in
the Boundary Conditions

We shall introduce

$$\zeta = \frac{\xi_1}{\sqrt{\eta_1}} \quad \text{and} \quad Y = \eta_1 \quad (29)$$

as new variable. With

$$T(\zeta, Y) = Z(\xi_1, \eta_1) \quad (30)$$

one calculates

$$\frac{\partial Z}{\partial \xi_1} = \frac{\partial T}{\partial \zeta} \frac{1}{\sqrt{Y}}, \quad \frac{\partial^2 Z}{\partial \xi_1^2} = \frac{\partial^2 T}{\partial \zeta^2} \frac{1}{Y}, \quad \frac{\partial Z}{\partial \eta_1} = -\frac{1}{2} \frac{\partial T}{\partial \zeta} \frac{\zeta}{Y} + \frac{\partial T}{\partial Y}$$

Hence (27) becomes

$$\frac{\partial T}{\partial \zeta} \left[a(Y) \sqrt{Y} - \frac{1}{2} b(Y) \zeta \right] + Y b(Y) \frac{\partial T}{\partial Y} = \sqrt{T} \frac{\partial^2 T}{\partial \zeta^2} \quad (31)$$

According to (28), the boundary conditions

$$T(\infty, Y) = 4, \quad T(0, Y) = 0 \quad (32)$$

must be satisfied.

Finally, a last simplification is obtainable by introduction of

$$\tau = \sqrt{\zeta} \quad \text{and} \quad \sigma = \sqrt{Y} \quad (33)$$

With

$$V(\tau, \sigma) = T(\zeta, Y) \quad (34)$$

one calculates

$$\frac{\partial T}{\partial \zeta} = \frac{\partial V}{\partial \tau} \frac{1}{2\tau}, \quad \frac{\partial T}{\partial Y} = \frac{\partial V}{\partial \sigma} \frac{1}{2\sigma}, \quad \frac{\partial^2 T}{\partial \zeta^2} = \frac{1}{4\tau} \left(\frac{1}{\tau} \frac{\partial^2 V}{\partial \tau^2} - \frac{1}{\tau^2} \frac{\partial V}{\partial \tau} \right)$$

equation (31), with

$$\left. \begin{aligned} a(Y)\sqrt{Y} &= a(\sigma^2)\sigma = A(\sigma) \\ b(Y) &= b(\sigma^2) = B(\sigma) \end{aligned} \right\} \quad (35)$$

becomes

$$\frac{\partial V}{\partial \tau} \frac{1}{2\tau} \left[A(\sigma) - \frac{B(\sigma)}{2} \tau^2 \right] + \sigma^2 B(\sigma) \frac{1}{2\sigma} \frac{\partial V}{\partial \sigma} = \sqrt{V} \frac{1}{4\tau} \left(\frac{1}{\tau} \frac{\partial^2 V}{\partial \tau^2} - \frac{1}{\tau^2} \frac{\partial V}{\partial \tau} \right)$$

or

$$\sqrt{V} \frac{\partial^2 V}{\partial \tau^2} + \frac{\partial V}{\partial \tau} \left[-2\tau A(\sigma) + B(\sigma)\tau^3 - \frac{1}{\tau} \sqrt{V} \right] = 2\tau^2 \sigma B(\sigma) \frac{\partial V}{\partial \sigma} \quad (36)$$

According to (32), the boundary conditions (fig. 6)

$$V(\infty, \sigma) = 4, \quad V(0, \sigma) = 0 \quad (37)$$

also must be satisfied.

The numerical treatment of the problem (equations (36) and (37)) can take place in exactly the same way as that of the special case of constant suction velocity, treated in detail in the next chapter.

II. THE CASE OF HOMOGENEOUS SUCTION

6. Specialization to Homogeneous Suction

Thus we put from now on

$$v_0(x) = v_0 \quad \text{with } v_0 < 0 \quad (38)$$

Then one obtains according to (18)

$$f(x_2) = -v_0 x_2 \quad (39)$$

according to (21)

$$F(x_2) = \sqrt{1 + v_0^2} \quad (40)$$

according to (20) and (22)

$$\eta_1 = \sqrt{1 + v_0^2} x_2, \quad x_2 = \frac{\eta_1}{\sqrt{1 + v_0^2}} \quad (41)$$

according to (25)

$$a(\eta_1) = v_0 \quad (42)$$

and according to (26)

$$b(\eta_1) = +\sqrt{1 + v_0^2} \quad (43)$$

The differential equation (24) is therefore transformed into

$$v_0 \frac{\partial Z}{\partial \xi_1} + \sqrt{1 + v_0^2} \frac{\partial Z}{\partial \eta_1} = \sqrt{Z} \frac{\partial^2 Z}{\partial \xi_1^2} \quad (44)$$

with the boundary conditions (28).

For further considerations, in order to become independent of v_0 , one introduces, slightly different from the procedure in the general case (paragraph 5), first as new independent coordinates

$$X = -a\xi_1 = -v_0\xi_1, \quad Y = \frac{a^2}{b} \eta_1 = \frac{v_0^2}{\sqrt{1 + v_0'^2}} \eta_1 \quad (45)$$

With

$$R(X, Y) = Z(\xi_1, \eta_1) \quad (46)$$

equation (44) then becomes

$$-\frac{\partial R}{\partial X} + \frac{\partial R}{\partial Y} = \sqrt{R} \frac{\partial^2 R}{\partial X^2} \quad (47)$$

according to (28) the boundary conditions

$$R(\infty, Y) = 4 = R(X, 0), \quad R(0, Y) = 0 \quad (48)$$

also must be satisfied.

In analogy to (29) one now uses

$$\zeta = \frac{X}{\sqrt{Y}} \quad \text{and} \quad Y \quad (49)$$

then according to (33) τ and σ as independent variables and thus obtains instead of (36) the differential equation

$$\sqrt{V} \frac{\partial^2 V}{\partial \tau^2} + \frac{\partial V}{\partial \tau} \left(2\tau\sigma + \tau^3 - \frac{1}{\tau} \sqrt{V} \right) = 2\tau^2\sigma \frac{\partial V}{\partial \sigma} \quad (50)$$

with the boundary conditions (37).

7. The Series Development in Powers of σ

Equation (50) is a nonlinear parabolic differential equation of the second order. Usually for problems of this kind (for instance for the equation of heat conduction) the values, for instance, of V are prescribed even for $\sigma = 0$; the reason that they are not prescribed here is probably that $\partial V / \partial \sigma$ contains the factor σ which, for $\sigma \rightarrow 0$, tends toward zero. Thus $\sigma = 0$ is a singularity of the differential equation. More important conclusions will result from this fact. In order to find out whether in spite of it the problem (equations (50) and (37)) can be regularly resolvable in the neighborhood of $\sigma = 0$, one sets up the series development

$$V(\tau, \sigma) = V_0(\tau) + V_1(\tau)\sigma + V_2(\tau)\sigma^2 + \dots \quad (51)$$

There follows from it

$$\sqrt{V} = \sqrt{V_0} + \frac{1}{2} \frac{V_1}{\sqrt{V_0}} \sigma + \left(\frac{1}{2} \frac{V_2}{\sqrt{V_0}} - \frac{1}{8} \frac{V_1^2}{\sqrt{V_0}^3} \right) \sigma^2 + \dots \quad (52)$$

Comparison of coefficients of the terms not containing σ in (50) gives

$$\sqrt{V_0} V_0'' + V_0' \left(\tau^3 - \frac{\sqrt{V_0}}{\tau} \right) = 0 \quad (53)$$

if the prime signifies the differentiation of V_0 with respect to the only variable τ . Aside from this nonlinear ordinary differential equation of the second order, according to (37) the boundary conditions

$$V_0(0) = 0, \quad V_0(\infty) = 4 \quad (54)$$

also have to be satisfied.

For the special case $v_0 = 0$ one has the problem of the plate in longitudinal flow without suction as it has been solved already by Blasius (reference 4). For this problem the corresponding quantities of which will always be designated by an asterisk is, according to (42) and (43),

$$a^*(\eta_1) = 0, \quad b^*(\eta_1) = 1 \quad (55)$$

so that (27) is transformed into

$$\frac{\partial Z^*}{\partial \eta_1} = \sqrt{Z^*} \frac{\partial^2 Z^*}{\partial \xi_1^2} \quad (56)$$

If one performs instead of (45) the identical transformation

$$X = \xi_1, Y = \eta_1, R^* = Z^* \quad (57)$$

equation (56) becomes instead of (47)

$$\frac{\partial R^*}{\partial Y} = \sqrt{R^*} \frac{\partial^2 R^*}{\partial X^2} \quad (58)$$

Furthermore one performs the same transformations as in paragraph 6 and obtains instead of (50)

$$\sqrt{v^*} \frac{\partial^2 v^*}{\partial \tau^2} + \frac{\partial v^*}{\partial \tau} \left(\tau^3 - \frac{1}{\tau} \sqrt{v^*} \right) = 2\tau^2 \sigma \frac{\partial v^*}{\partial \sigma} \quad (59)$$

with the boundary conditions (37). This problem has a solution

$$v^* = v^*(\tau) \quad (60)$$

which is independent of σ and determined by

$$\sqrt{v^*} v^{*''} + v^{*'} \left(\tau^3 - \frac{1}{\tau} \sqrt{v^*} \right) = 0, v^*(0) = 0, v^*(\infty) = 4 \quad (61)$$

This is the well-known solution of Blasius.

Comparison of (61) on one hand with (53) and (54) on the other hand shows

$$V_0(\tau) = v^*(\tau) \quad (62)$$

One knows (and can verify it immediately by power series development for (61)) that

$$\sqrt{v_0(\tau)} = \sqrt{v^*(\tau)} = a_1\tau + a_2\tau^2 + \dots \quad (63)$$

thus

$$v_0(\tau) = v^*(\tau) = a_1^2\tau^2 + 2a_1a_2\tau^3 + \dots \quad (64)$$

One now determines the coefficient $v_1(\tau)$ in the development (51). Comparison of coefficients of the power σ in (50) gives

$$\sqrt{v_0} v_1'' + \frac{1}{2} \frac{v_1}{\sqrt{v_0}} v_0'' + v_1' \left(\tau^3 - \frac{1}{\tau} \sqrt{v_0} \right) + v_0' \left(2\tau - \frac{1}{\tau} \frac{1}{2} \frac{v_1}{\sqrt{v_0}} \right) = 2\tau^2 v_1$$

or

$$\sqrt{v_0} v_1'' + v_1' \left(\tau^3 - \frac{1}{\tau} \sqrt{v_0} \right) + v_1 \left(\frac{1}{2} \frac{v_0''}{\sqrt{v_0}} - 2\tau^2 - \frac{1}{2\tau} \frac{v_0'}{\sqrt{v_0}} \right) = -2\tau v_0' \quad (65)$$

According to (37), because of (54), the boundary conditions

$$v_1(0) = 0 = v_1(\infty) \quad (66)$$

also have to be satisfied. If one uses for $\sqrt{v_1(\tau)}$ in analogy to (63) the expression

$$\sqrt{v_1(\tau)} = b_1\tau + b_2\tau^2 + \dots, \quad v_1 = b_1^2\tau^2 + 2b_1b_2\tau^3 + \dots \quad (67)$$

comparison of coefficients for the coefficient of τ necessarily gives

$$a_1 2b_1^2 - 2b_1^2 a_1 + b_1^2 \left(\frac{1}{2} \frac{2a_1^2}{a_1} - \frac{1}{2} \frac{2a_1^2}{a_1} \right) = 0$$

since one parameter must be kept available for the fulfilment of the second boundary condition (66). For the further coefficients b_ν with $\nu > 1$ one obtains qualifying equations. The further coefficients $V_\mu(t)$ with $\mu > 1$ in (51) also could be determined in this manner; however, this way would be too troublesome. We shall therefore choose another method.

The fundamentally important result of this paragraph is the knowledge of $V(\tau, 0) = V_0(\tau) = V^*(\tau)$. According to it, any method of profile continuation (as it was developed for instance by L. Prandtl (reference 8), H. Görtler (reference 9) and K. Schröder (reference 10) is applicable for the treatment of the problem (50), (37). However, if the expenditure of calculation time is not considerably increased, these methods have too little accuracy, particularly in the derivatives with respect to τ . Since, in the interest of a stability investigation for not too large values of σ , (see references 11 and 12) and, for the stability investigation of the present problem, (reference 13) the second derivatives of $V(\tau, \sigma)$ with respect to τ also are required, the application of the two methods to be described below is justified: though they require more time expenditure than the methods described in references 8, 9, and 10, they have the advantage of giving the derivatives of the velocity component u with respect to the transverse direction with a relatively high accuracy. But also aside from the stability investigation the present problem is of such basic significance that the desire for greater accuracy of the velocity distribution is sufficiently justified. Inversely, the two methods in question can of course be used for continuation of arbitrarily prescribed velocity profiles.

8. A First Approximation Method

$V(\tau, \sigma) = V(\tau)$ will be approximately calculated for constant σ according to (50) under the assumption that $V(\tau, \sigma - k) = V(\tau)$ is known; for $\sigma - k = 0$ this assumption is correct. One replaces in (50) $\partial V / \partial \sigma$ approximately by the difference quotient $(V - \bar{V})/k$; if k is small, one may furthermore put $V \approx \bar{V}$; this will be done in (50) everywhere V appears in a power superior to 1. One then obtains from (50) the following equation for the linear approximation

$$\sqrt{\bar{V}} V'' + \left(2\tau\sigma + \tau^3 - \frac{1}{\tau} \sqrt{\bar{V}} \right) V' = \frac{2\tau^2\sigma}{k} (V - \bar{V}) \quad (68)$$

Furthermore, according to (37) the boundary conditions

$$V(0) = 0, V(\infty) = 4 \quad (69)$$

have to be satisfied. The considerations of paragraph 7 lead to the series development in powers

$$\sqrt{V(\tau)} = c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4 + \dots \quad (70)$$

thus

$$V(\tau) = d_2\tau^2 + d_3\tau^3 + d_4\tau^4 + d_5\tau^5 + \dots \quad (71)$$

assuming as analogous

$$\left. \begin{aligned} \sqrt{\bar{V}} &= a_1\tau + \dots + a_4\tau^4 + \dots \\ \bar{V} &= b_2\tau^2 + \dots + b_5\tau^5 + \dots \end{aligned} \right\} \quad (72)$$

in the case $\sigma - k = 0$ this expression according to (63) and (64) is justified. The a_ν and b_ν are connected by the formulas

$$\left. \begin{aligned} b_2 &= a_1^2 & a_1 &= \sqrt{b_2} \\ b_3 &= 2a_1a_2 & a_2 &= b_3:2a_1 \\ b_4 &= 2a_1a_3 + a_2^2 & a_3 &= (b_4 - a_2^2):2a_1 \\ b_5 &= 2a_1a_4 + 2a_2a_3 & a_4 &= (b_5 - 2a_2a_3):2a_1 \\ & \vdots & & \vdots \end{aligned} \right\} \quad (73)$$

Comparison with the coefficients in (68) yields for calculation of the d_v the formulas

$$\tau^2 \left\| \begin{array}{l} 2 \times 3a_1d_3 + 1 \times 2a_2d_2 + 2d_2(2\sigma - a_2) - 3d_3a_1 = 0 \end{array} \right.$$

$$\tau^3 \left\| \begin{array}{l} 3 \times 4a_1d_4 + 2 \times 3a_2d_3 + 1 \times 2a_3d_2 - 2a_3d_2 + 3d_3(2\sigma - a_2) \end{array} \right.$$

$$- 4a_1d_4 = 0$$

$$\tau^4 \left\| \begin{array}{l} 4 \times 5a_1d_5 + 3 \times 4a_2d_4 + 2 \times 3a_3d_3 + 1 \times 2a_4d_2 + 2d_2(1 - a_4) \end{array} \right.$$

$$- 3a_3d_3 + 4d_4(2\sigma - a_2) - 5a_1d_5 = (d_2 - b_2)\sigma/k$$

⋮

Further calculation yields the following formulas for determination of d_v :

$$\left. \begin{array}{l} d_3 = -\frac{4\sigma d_2}{3a_1} \\ d_4 = -\frac{3d_3(a_2 + 2\sigma)}{8a_1} \\ d_5 = -\frac{8d_4(a_2 + \sigma) - 3d_3a_3 - 2d_2(1 - \sigma/k) - 2b_2\sigma/k}{15a_1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right\} (74)$$

The further numerical treatment will be discussed in paragraph 12.

9. Variables Suitable for Tabulation

We shall now consider what dimensionless variables are most suitable as characteristics and what connection they have with the coordinates τ and σ forming the basis of our calculation. We shall in

$$V(\tau, \sigma) = U^2(\tau, \sigma) \quad (75)$$

again introduce the original variables. According to (33), (49), and (45)

$$\begin{aligned} U(\tau, \sigma) &= U(\sqrt{\xi}, \sqrt{Y}) = U\left(\sqrt{\frac{X}{\sqrt{Y}}}, \sqrt{Y}\right) \\ &= U\left(\sqrt{\frac{v_0 \xi_1}{v_0 \sqrt{\frac{\eta_1}{\sqrt{1+v_0^2}}}}}, -v_0 \sqrt{\frac{\eta_1}{\sqrt{1+v_0^2}}}\right) \end{aligned}$$

thus

$$U(\tau, \sigma) = U\left(\sqrt{\frac{\xi_1 \sqrt[4]{1+v_0^2}}{\sqrt{\eta_1}}}, \frac{-v_0 \sqrt{\eta_1}}{\sqrt[4]{1+v_0^2}}\right) \quad (76)$$

Further one obtains according to (41), (39), and (20)

$$U(\tau, \sigma) = U\left(\sqrt{\frac{\psi_2 + v_0 x_2}{\sqrt{x_2}}}, -v_0 \sqrt{x_2}\right) \quad (77)$$

From (19) one concludes with (39)

$$y_2 = \int_{-v_0 x_2}^{\psi_2} \frac{d\psi_2}{U\left(\sqrt{\frac{\psi_2 + v_0 x_2}{x_2}}, -v_0 \sqrt{x_2}\right)} \quad (78)$$

for constant x_2 . Here

$$\tau^2 = \frac{\psi_2 + v_0 x_2}{x_2}, \quad 2\tau d\tau = \frac{d\psi_2}{\sqrt{x_2}} \quad (79)$$

we shall introduce as new integration variable

$$y_2 = 2\sqrt{x_2} \int_0^\tau \frac{\tau d\tau}{U(\tau, \sigma)} \quad (80)$$

for constant σ . According to (16), (14) and the following transformation equations as well as (75) one concludes

$$U(\tau, \sigma) = U_2(\psi_2, x_2) = u_2(x_2, y_2) \quad (81)$$

According to (9) one obtains further

$$u_1(x_1, y_1) = \frac{U_0}{2} u_2\left(\frac{2x_1}{U_0}, y_1\right) = \frac{U_0}{2} U(\tau, \sigma) \quad (82)$$

If one, furthermore, takes (6) into consideration, one has between the original coordinates x, y and the end coordinates τ, σ the connection

$$\sigma = -v_0 \sqrt{x_2} = -v_0 \sqrt{\frac{2}{U_0} x_1} = -v_0 \sqrt{\frac{2}{U_0} \frac{x}{v}} \quad (83)$$

$$\left. \begin{aligned} \tau &= \sqrt{\frac{\psi_2 + v_0 x_2}{\sqrt{x_2}}}, \quad y = v y_1 = v y_2 = 2v \sqrt{x_2} \int_0^\tau \frac{\tau d\tau}{U(\tau, \sigma)} \\ &= \frac{2\sigma v}{-v_0} \int_0^\tau \frac{\tau d\tau}{U(\tau, \sigma)} \end{aligned} \right\} \quad (84)$$

Due to (84) and (83),

$$y^* = \frac{-v_0 y}{v} = 2\sigma \int_0^\tau \frac{\tau d\tau}{U(\tau, \sigma)} \quad (85)$$

and

$$\sqrt{\xi} = \frac{-v_0}{\sqrt{U_0}} \sqrt{\frac{x}{v}} = \frac{-v_0}{U_0} \sqrt{\frac{U_0 x}{v}} = c_Q \sqrt{\frac{U_0 x}{v}} = \frac{1}{\sqrt{2}} \sigma$$

are therefore advisable as dimensionless variables. ($c_Q = -v_0/U_0$) is the so-called mass coefficient of the suction, defined by $Q = -v_0 b l = c_Q U_0 b l$, with Q signifying the suction quantity, b and l width and length, respectively, of the plate section considered.

10. The Characteristic Boundary Layer Parameters

As quantities particularly characteristic for the boundary layer one defines the displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy \quad (86)$$

thus with (85)

$$\frac{-v_o \delta^*}{\nu} = \int_0^{\infty} \left(1 - \frac{u}{U_o}\right) dy^* = \sigma \int_0^{\infty} \left[1 - \frac{U(\tau, \sigma)}{2}\right] \frac{\tau d\tau}{U(\tau, \sigma)} \quad (87)$$

the momentum thickness

$$\vartheta = \int_0^{\infty} \frac{u}{U_o} \left(1 - \frac{u}{U_o}\right) dy \quad (88)$$

thus

$$\frac{-v_o \vartheta}{\nu} = \int_0^{\infty} \frac{u}{U_o} \left(1 - \frac{u}{U_o}\right) dy^* = \sigma \int_0^{\infty} \left[1 - \frac{U(\tau, \sigma)}{2}\right] \tau d\tau \quad (89)$$

the form parameter

$$\frac{\delta^*}{\vartheta} = \frac{-v_o \delta^*}{\nu} \div \frac{-v_o \vartheta}{\nu} \quad (90)$$

and, finally, the wall shearing stress (ρ = density, $\rho\nu = \mu$)

$$\tau_o = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \quad (91)$$

thus

$$\frac{\tau_o \delta^*}{\mu U_o} = \frac{-v_o \delta^*}{\nu} \left[\frac{\partial}{\partial y^*} \left(\frac{u}{U_o}\right) \right]_{y^*=0} \quad (92)$$

11. The Blasius Solution without Suction

Since the Blasius solution forms the starting point for the entire calculation, we shall first sketch its determination in our coordinates. For the numerical calculation of the solution $V^*(\tau)$ of the problem (61) the method of Adams was used, and in particular the extrapolation method (cf. G. Schulz (reference 14), Nr. 108). For practical calculation with the calculating machine, the following formula (symbols as in Schulz' report) is more convenient than the formula (2) (cf. the equation (1) on p. 116, Schulz) which uses the differences of the function $f(x,y)$

$$\begin{aligned} \mathfrak{V}_{v+1} - \mathfrak{V}_v &= h(2.291667\mathfrak{f}_v - 2.458333\mathfrak{f}_{v-1} \\ &+ 1.541667\mathfrak{f}_{v-2} - 0.375\mathfrak{f}_{v-3}) \end{aligned} \quad (93)$$

equation (93) operates only with the values of the function $f(x,y)$ instead of using their differences. This formula was applied with the width of interval $h = 0.05$ to the system of differential equations following from (61)

$$W^{*'} = -\frac{W^*\left(\tau^3 - \frac{1}{\tau}\sqrt{V^*}\right)}{\sqrt{V^*}}, \quad V^{*'} = W^* \quad (94)$$

The necessary four initial values were determined by series development. If one uses for $\sqrt{V^*}$ and V^* the expression (72), the result obtained by comparison of coefficients for (61) is

$$\begin{aligned} a_2 = a_3 = a_5 = a_6 = 0, \quad a_4 &= \frac{1}{15} \\ b_3 = b_4 = b_6 = b_7 = 0, \quad b_2 &= a_1^2, \quad b_5 = 2a_1a_4 \end{aligned} \quad (95)$$

For determination of the as yet undetermined coefficient a_1 or b_2 the value

$$f_0''(0) = 1.328242 \quad (96)$$

was used which can be found in reference 15, p. 551. According to Howarth, p. 550, formula (2), the variable is

$$\eta = \frac{y}{2} \sqrt{\frac{U_0}{\nu x}} \quad (97)$$

in our symbols

With $v_0 = 0$ equation (76) is, when (57) is taken into consideration, transformed into

$$U^*(\tau) = U^* \left(\sqrt{\frac{\xi_1}{\eta_1}} \right) = \sqrt{v^*(\tau)} \quad (98)$$

equation (82) becomes

$$u_1^*(x_1, y_1) = \frac{U_0}{2} U^*(\tau) \quad (99)$$

and (84)

$$\tau = \sqrt{\frac{\psi_2}{x_2}} \quad y = \nu y_1 = \nu y_2 = 2\nu \sqrt{x_2} \int_0^\tau \frac{\tau d\tau}{U^*(\tau)} = 2 \sqrt{\frac{2\tau x}{U_0}} \int_0^\tau \frac{\tau d\tau}{U^*(\tau)} \quad (100)$$

Hence (97) is transformed into

$$\eta = \sqrt{2} \int_0^\tau \frac{\tau d\tau}{U^*(\tau)} \quad (101)$$

According to Howarth (reference 15, p. 548) $f'_0(\eta)$ is identical with our velocity U^* , though as function of η . Thus one has

$$f_0''(\eta) = \frac{dU^*(\tau)}{d\tau} \frac{d\tau}{d\eta}$$

or, taking (101) into consideration,

$$\frac{dU^*(\tau)}{d\tau} = f_0'''(\eta) \frac{d\eta}{d\tau} = f_0'''(\eta)\sqrt{2} \frac{\tau}{U^*(\tau)} \quad (102)$$

If one puts here $\tau = \eta = 0$, one obtains, taking the development (72) into consideration

$$a_1^2 = b_2 = f_0'''(0)\sqrt{2} \quad (103)$$

For improvement of the values attained by means of Adams' extrapolation formula (93) the interpolation formula of Adam (Schulz, reference 14, p. 121, last formula (8)) was used which, when the differences are eliminated, appears as follows:

$$y_m^{(k+1)} - y_{m-1}^{(k+1)} = h \left[0.375f_m^{(k)} + 0.791667f_{m-1}^{(k)} - 0.208334f_{m-2}^{(k)} + 0.041667f_{m-3}^{(k)} \right] \quad (104)$$

The numerical values thus calculated of

$$\frac{u}{U_0}, \frac{1}{U_0} \frac{\partial u}{\partial \eta^*} \quad \text{and} \quad \frac{1}{U_0} \frac{\partial^2 u}{\partial \eta^{*2}}$$

are compiled at the beginning of table 1 as function of

$$\eta^* = y \sqrt{U_0/\nu_x}$$

The characteristic boundary layer parameters are to be defined as in 10. If one introduces as dimensionless variable (cf. also (100))

$$\eta^* = y \sqrt{\frac{U_0}{\nu x}} = \sqrt{2} \int_0^\tau \frac{\tau d\tau}{U^*(\tau)} \quad (105)$$

(this is the quantity 2η used by Howarth, cf. (97) and (101)), there follows according to (86)

$$\delta^* \sqrt{\frac{U_0}{\nu x}} = \int_0^\infty \left(1 - \frac{u^*}{U_0}\right) d\eta^* = \sqrt{2} \int_0^\infty \left(1 - \frac{U^*(\tau)}{2}\right) \frac{\tau d\tau}{U^*(\tau)} \quad (106)$$

In analogous manner one obtains, according to (88), for the momentum thickness

$$\delta \sqrt{\frac{U_0}{\nu x}} = \sqrt{2} \int_0^\infty \left(1 - \frac{U^*(\tau)}{2}\right) \tau d\tau \quad (107)$$

The form parameter again is

$$\frac{\delta^*}{\delta} = \frac{\delta^* \sqrt{\frac{U_0}{\nu x}}}{\delta \sqrt{\frac{U_0}{\nu x}}} \quad (108)$$

and the wall shearing stress results, according to (91), as

$$\frac{\tau_0}{\mu} \frac{\delta^*}{U_0} = \delta^* \sqrt{\frac{U_0}{\nu x}} \left. \frac{\partial \left(\frac{u}{U_0}\right)}{\partial \eta^*} \right|_{\eta^*=0} \quad (109)$$

The following numerical values result:

$$\left. \begin{aligned} \delta^* \sqrt{\frac{U_0}{\nu x}} &= 1.7207, & \frac{\tau_0}{\mu} \frac{\delta^*}{U_0} &= 0.5714 \\ \delta \sqrt{\frac{U_0}{\nu x}} &= 0.6641, & \frac{\delta^*}{\delta} &= 2.591 \end{aligned} \right\} \quad (110)$$

12. The Numerical Determination of the Approximate
Solution $V(\tau, \sigma)$ According to the First Method

First, a solution of (68) and (69) was determined for $\sigma = 0.1$ hence, one had to put

$$\bar{V}(\tau) = V(\tau, 0) = V^*(\tau)$$

The series development described in paragraph 8 can be performed according to the formulas given there, only the coefficient d_2 (and hence c_1) remains undetermined. Thus one has to assume first an arbitrary value of d_2 and has to determine it in the end from the second condition (69). It will be useful to choose $d_2 \sim b_2$. The calculation was made in a manner similar to one applied by Hans Joachim Luckert (reference 16). Two different values b_2 (let us assume b_{21} and b_{22}) were selected, the corresponding values of $V(\tau)$ for $\tau = 0; 0.05; 0.1; 0.15$ were calculated according to the series (71). Then the two solutions, to be called $V_1(\tau)$ and $V_2(\tau)$, were followed up further by means of Adams' method as in paragraph 11 (cf. (93) and (104)); the two values b_{21} and b_{22} were selected in such a manner that $V_1(\tau) > \bar{V}(\tau)$ and $V_2(\tau) < \bar{V}(\tau)$ are valid for small values of τ . As soon as $V_1(\tau)$ or $V_2(\tau)$ at a point τ_0 exceeded the function $V(\tau)$ or deviated from it by more than 1, two new functions $V_3(\tau)$ and $V_4(\tau)$ were introduced by

$$V_3(\tau) = mV_1(\tau) + nV_2(\tau), \quad V_4(\tau) = \mu V_1(\tau) + V_2(\tau) \quad (111)$$

with

$$\left. \begin{aligned} m &= \frac{V_3(\tau_0) - V_2(\tau_0)}{V_1(\tau_0) - V_2(\tau_0)}, \quad n = 1 - m \\ \mu &= \frac{V_4(\tau_0) - V_2(\tau_0)}{V_1(\tau_0) - V_2(\tau_0)}, \quad \nu = 1 - \mu \end{aligned} \right\} \quad (112)$$

$V_3(\tau_0) > \bar{V}(\tau_0)$ and $V_4(\tau_0) < \bar{V}(\tau_0)$ were selected so that behind τ_0 again $V_3(\tau)$ lies above, $V_4(\tau)$ below $\bar{V}(\tau)$. After calculation of the starting values of V_3 and V_4 required for the Adams method and of the corresponding derivatives at the points $\tau_0; \tau_0 - 0.05; \tau_0 - 0.1;$ and $\tau_0 - 0.15$ according to (111), V_3 and V_4 may be calculated further according to Adams. Of course $V_3(\tau_0) = V_1(\tau_0)$ or $V_4(\tau_0) = V_2(\tau_0)$ may be selected if that seems suitable. Possibly the pair $V_3(\tau), V_4(\tau)$ also must be replaced in the same manner by $V_5(\tau), V_6(\tau)$, etc. The calculation was carried through up to a pair of functions $V_{2n-1}(\tau), V_{2n}(\tau)$ until it became evident that for instance $V_{2n-1}(\tau)$ with the assumption $V_{2n-1}(\tau_1) = 4$ did not any longer show a tendency to deviate from the value 4 to a noteworthy degree. This was shown to occur at $\tau_1 = 3;$ for the larger σ values this point shifted to the left (cf. paragraph 18). The desired solution of (68) for $\sigma = 0.1$ then is

$$V(\tau; 0.1; 0.1) = V_{2n-1}(\tau) \quad (113)$$

This solution was calculated for all τ values, taking the formulas (111) and (112) into consideration.

However, the accuracy of this $V(\tau; 0.1; 0.1)$ is not yet sufficient. Thus $V(\tau; 0.1; 0.05)$ also was calculated, by subdividing the interval $\sigma = 0$ to $\sigma = 0.1$ into two partial intervals of half the width; the numerical treatment was the same.

In order to obtain an estimate of the accuracy, the σ interval of the width 0.1 finally was divided into four equal parts and thus $V(\tau; 0.1; 0.025)$ obtained. By linear extrapolation one obtained from

$$V(\tau; 0.1; 0.1) \text{ and } V(\tau; 0.1; 0.05)$$

$$V_1(\tau; 0.1) = 2V(\tau; 0.1; 0.05) - V(\tau; 0.1; 0.1) \quad (114)$$

furthermore by quadratic extrapolation from

$$V(\tau; 0.1; 0.025), V(\tau; 0.1; 0.05) \text{ and } V(\tau; 0.1; 0.1)$$

$$V_2(\tau; 0.1) = 2.666667V(\tau; 0.1; 0.025) \quad (115)$$

$$- 2V(\tau; 0.1; 0.05) + 0.333333V(\tau; 0.1; 0.1)$$

The functions (114) and (115) differed at most by one unit of the third decimal, their first and second derivatives with respect to τ at most by 2 or 6 units of the third decimal. Finally one put

$$V(\tau; 0.1) \equiv V_2(\tau; 0.1) \quad (116)$$

In the same way one calculated

$$V(\tau, \sigma) \text{ for } \sigma = 0.2; 0.3; \dots; 1.0$$

using

$$\bar{V}(\tau) = V(\tau; \sigma - 0.1)$$

etc.; however, one always puts

$$V(\tau, \sigma) \equiv V_1(\tau; \sigma) \text{ for } \sigma > 0.1 \quad (117)$$

The sub-division of the σ interval of the width 0.1 into four equal parts was not undertaken. Actually, (verified by the sketches of the curves), the variation of $V(\tau, \sigma)$ and its derivatives as well as the velocity variation calculated from it with its derivatives becomes more and more uniform with increasing σ values.

13. A Second Approximation Method

Following, another approximation method will be described which yields results much more rapidly; it requires only about one third of the time expenditure of the former method and operates with at least the same accuracy. The problem to be solved is (50), (37). By interpolation one determines

$$\frac{\partial V(\tau, \sigma)}{\partial \sigma} = \frac{1}{k} \left[-0.33333V(\tau, \sigma - 3k) + 1.5V(\tau, \sigma - 2k) \right. \\ \left. - 3V(\tau, \sigma - k) + 1.83333V(\tau, \sigma) \right] \quad (117)$$

$V(\tau, \sigma)$ is calculated under the assumption that all $V(\tau, \sigma - nk)$ for $n = 1, 2, 3, 4$ are known. (Actually this method could, therefore, have been used already in the calculation of $V(\tau; 0.4)$). For $V(\tau, \sigma)$ one first determined by extrapolation the approximate value

$$V^*(\tau, \sigma) = -V(\tau, \sigma - 4k) + 4V(\tau, \sigma - 3k) - 6V(\tau, \sigma - 2k) \\ + 4V(\tau, \sigma - k) \quad (118)$$

The (50) is approximately replaced by

$$\frac{\partial^2 V}{\partial \tau^2} + \frac{\partial V}{\partial \tau} \left(\frac{2\tau\sigma + \tau^3}{\sqrt{V^*}} - \frac{1}{\tau} \right) = \frac{2\tau^2\sigma}{\sqrt{V^*}} \frac{\partial V}{\partial \sigma} \quad (119)$$

If one puts according to (117)

$$g(\tau) = \frac{2\tau^2\sigma}{\sqrt{V^*k}} \left[-0.33333V(\tau, \sigma - 3k) + 1.5V(\tau, \sigma - 2k) + 3V(\tau, \sigma - k) \right] \quad (120)$$

equation (119) is transformed into ²

$$\frac{\partial^2 V}{\partial \tau^2} + \frac{\partial V}{\partial \tau} \left(\frac{2\tau\sigma + \tau^3}{\sqrt{V^*}} - \frac{1}{\tau} \right) - 1.83333 \frac{2\tau^2\sigma}{k\sqrt{V^*}} V = g(\tau) \quad (121)$$

This equation is solved in the manner described in paragraph 12 with the boundary conditions

$$V(0) = 0; V(\infty) = 4 \quad (122)$$

Split into a system of differential equations of the first order, it will appear as follows:

$$\begin{aligned} V' &= W \\ W' &= -W \left(\frac{2\tau\sigma + \tau^3}{\sqrt{V^*}} - \frac{1}{\tau} \right) + 1.83333 \frac{2\tau^2\sigma}{k\sqrt{V^*}} V + g(\tau) \end{aligned} \quad (123)$$

²One also could have determined $\partial V(\tau, \sigma) / \partial \sigma$ instead of according to (117) by interpolation from

$$V(\tau, \sigma - k), V(\tau, \sigma - 2k), V(\tau, \sigma - 3k), V(\tau, \sigma - 4k)$$

by extrapolation. In place of (121) one would then have obtained for $V(\tau, \sigma)$ a linear differential equation of the second order in which the term with V is missing and which thus can be solved by quadratures (Simpson's rule). This method may perhaps seem more convenient to many readers.

For determination of the starting values for Adams' method one again sets up the series developments (70), (71) and analogously

$$\sqrt{V^*} = c_1 * \tau + c_2 * \tau^2 + \dots + c_5 * \tau^5 + \dots \quad (124)$$

$$V^* = d_2 * \tau^2 + \dots + d_6 * \tau^6 + \dots \quad (125)$$

According to (118)

$$d_v^* = -d_v(\sigma - 4k) + 4d_v(\sigma - 3k) - 6d_v(\sigma - 2k) + 4d_v(\sigma - k) \quad (126)$$

Furthermore according to (120)

$$\sqrt{V^*}g(\tau) = A_4 \tau^4 + \dots \quad (127)$$

with

$$A_4 = 2\sigma \frac{1}{k} \left[-0.33333b_2(\sigma - 3k) + 1.5b_2(\sigma - 2k) - 3b_2(\sigma - k) \right] \quad (128)$$

The comparison of coefficients is made, instead of with (121), to better advantage with

$$\sqrt{V^*} \frac{\partial^2 v}{\partial \tau^2} + \frac{\partial v}{\partial \tau} \left(2\tau\sigma + \tau^3 - \frac{1}{\tau} \sqrt{V^*} \right) - 1.83333 \frac{2\tau^2\sigma}{k} v = \sqrt{V^*}g(\tau) \quad (129)$$

One obtains

$$\begin{aligned}
 d_3(\sigma) &= -\frac{4\sigma d_2(\sigma)}{3c_1^*} \\
 d_4(\sigma) &= -\frac{(3c_2^* + 6\sigma)d_3(\sigma)}{8c_1^*} \\
 d_5(\sigma) &= \frac{A_4 - \left(2 - 3.66667 \frac{\sigma}{k}\right)d_2(\sigma) + 3c_3^*d_3(\sigma) + 8(\sigma + c_2^*)d_4(\sigma)}{15c_1^*} \\
 &\quad \cdot \\
 &\quad \cdot
 \end{aligned}
 \tag{130}$$

It is useful to select for the arbitrary $d_2(\sigma)$

$$d_2(\sigma) \approx d_2^* \tag{131}$$

The first calculation according to this method was carried out for $\sigma = 1.2$ and $k = 0.2$; the maximum difference between V and V^* amounted to less than 5 units of the fourth decimal. Nevertheless one must not think the application of Adams' method superfluous; it should be recalled that one obtains in this manner

$$\partial V / \partial \tau \text{ and } \partial^2 V / \partial \tau^2$$

with appropriate accuracy which is not well possible by means of extrapolation. And some time, of course, this good agreement between V and V^* does have to stop. For $\sigma = 1.6$, for instance, the difference was already one unit of the third decimal.

The following σ values were treated according to the method described above:

$$\sigma = 1.2; 1.4; 1.6; 2.0; 2.4; 3.2; 4.0$$

14. The Calculation of the Reduced Velocity and Its Derivatives

According to (75) and (82) one finds immediately

$$\frac{u}{U_0} = \frac{1}{2} \sqrt{V(\tau)} = \frac{1}{2} U(\tau) \quad (132)$$

first as a function of τ . The quantity y^* desired as independent variable instead of τ was calculated according to (85) by means of Simpson's rule. Furthermore one obtains, again by means of (85),

$$\frac{1}{U_0} \frac{\partial u}{\partial y^*} = \frac{1}{U_0} \frac{\partial u}{\partial \tau} \frac{\partial \tau}{\partial y^*} = \frac{1}{2} \frac{W(\tau)}{2\sqrt{V(\tau)}} \frac{\sqrt{V(\tau)}}{2\sigma\tau} = \frac{W(\tau)}{8\sigma\tau} \quad (133)$$

and according to (71)

$$\frac{1}{U_0} \frac{\partial u}{\partial y^*} = \frac{d_2}{4\sigma} \quad \text{for } \tau = 0 \quad (134)$$

In the same way one calculates

$$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}} = \frac{\tau W'(\tau) - W(\tau)}{8\sigma\tau^2} \frac{\sqrt{V(\tau)}}{2\sigma\tau} = \frac{1}{16\sigma^2} \frac{(\tau W' - W)\sqrt{V}}{\tau^3} \quad (135)$$

and

$$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}} = \frac{1}{16\sigma^2} 3c_1 d_3 \quad \text{for } \tau = 0 \quad (136)$$

The quantity $W'(\tau)$ appearing in (135) was already determined in the calculation according to Adams' method.

Table 1 added, for the various values of

$$\xi = 1/2\sigma^2$$

the calculated numerical values of

$$u/U_0, U_0^{-1} \frac{\partial u}{\partial y^*} \text{ and } U_0^{-1} \frac{\partial^2 u}{\partial y^{*2}}$$

as a function of y^* including the Blasius profile for $\xi = 0$ and the asymptotic suction profile (cf. (5)). Concerning the accuracy of all tables it should be noted that the last decimal given cannot always be fully guaranteed; particularly for smaller values of the independent variable small inaccuracies in the last decimal may occur³. In the following way one obtains another estimate of the accuracy of the numerical material. Fulfilment of Prandtl's boundary layer equation (2) for $y = 0$, that is at the plate, yields the result that there must be

$$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}} = -\frac{1}{U_0} \frac{\partial u}{\partial y^*} \text{ for } y^* = 0$$

In the approximate calculation this condition was not used.

Fig. 7 shows a sketch of u/U_0 as a function of

$$y^* = \nu_0 y / U_0$$

The velocity distributions approach for large ξ the asymptotic suction profile ($\xi = \infty$). In fig. 8 the curves u/U_0 against y/δ^* are illustrated with a shifted starting point. One can see that these curves hardly differ in shape. The curves of

$$\delta^*/U_0^{-1} \frac{\partial u}{\partial y}$$

illustrated in fig. 9 in the same manner, have a slightly more varied aspect, whereas

$$\frac{\partial^2 u}{\partial y^2}$$

³The numerical calculations were verified by Mr. H. Schaefer on the basis of the so-called momentum control

$$\frac{\tau_0(x)}{\rho} = U_0^2 \frac{d\delta}{dx} - \nu_0 U_0$$

it proved useful to integrate this equation in the x direction. Agreement according to the accuracy indicated above resulted.

(fig. 10) for the different values of the parameter ξ shows an entirely different shape.

The characteristic boundary layer parameters

$$-v_0 \delta^*/v \quad \text{and} \quad -v_0 \delta/v$$

result from (87) and (89), respectively, by means of Simpson's rule. Thereafter one may calculate immediately the form parameter δ^*/δ according to (90) and, according to (92), easily

$$\tau_0 \delta^*/\mu U_0$$

With (85) taken into consideration, there follows immediately from (87) and (106) or (89) and (107), respectively,

$$\left. \frac{d\left(-\frac{\delta^* v_0}{v}\right)}{d\sqrt{\xi}} \right|_{\sigma=0} = \delta^* \left. \sqrt{\frac{U_0}{\nu x}} \right|_{\text{Blasius}} = 1.7207 \quad (137)$$

and

$$\left. \frac{d\left(-\frac{\delta v_0}{v}\right)}{d\sqrt{\xi}} \right|_{\sigma=0} = \delta \left. \sqrt{\frac{U_0}{\nu x}} \right|_{\text{Blasius}} = 0.6641 \quad (138)$$

the numerical values were taken from (110).

The four characteristic constants are to be found in table 2 as function of ξ and $\sqrt{\xi}$, respectively. Fig. 11 gives an illustration by means of a sketch; the tangents at the starting point according to equations (137) and (138) are plotted there.

15. The Stream Line Pattern

As stream function Ψ we shall introduce the stream function $\Psi(\xi, y^*)$ with the use of the dimensionless coordinates; it is defined by

$$\frac{\partial \psi}{\partial \xi} = -\frac{v}{-v_0}, \quad \frac{\partial \psi}{\partial y^*} = \frac{u}{U_0} \quad (139)$$

For that purpose one puts

$$\psi = \frac{1}{2} (-v_0) \psi_2 \quad (140)$$

Actually, according to (83) and (85),

$$\xi = \frac{1}{2} (-v_0)^2 x_2 \quad (141)$$

hence according to (140), (13), and (9),

$$\frac{\partial \psi}{\partial \xi} = \frac{1}{2} (-v_0) \frac{\partial \psi_2}{\partial x_2} \frac{2}{(-v_0)^2} = -\frac{v}{-v_0}$$

furthermore according to (84) and (85)

$$y^* = (-v_0) y_2 \quad (142)$$

thus according to (140), (13), and (8)

$$\frac{\partial \psi}{\partial y^*} = \frac{1}{2} (-v_0) \frac{\partial \psi_2}{\partial y_2} \frac{1}{-v_0} = \frac{u}{U_0}$$

The numerical calculation of the function ψ may be carried out as follows: From (79) one concludes

$$\psi_2 = \sqrt{x_2} \tau^2 - v_0 x_2$$

with (83) there results

$$\psi_2 = \frac{\sigma}{-v_0} \tau^2 + \frac{\sigma^2}{-v_0}$$

thus according to (140)

$$\psi = \frac{1}{2} \sigma (\tau^2 + \sigma) \quad (143)$$

This value corresponds to

$$\xi = \frac{1}{2} \sigma^2 \quad \text{and} \quad y^*(\tau, \sigma) \quad (144)$$

the latter has to be calculated according to (85).

According to the tables calculated in this way the pattern of the stream lines $\psi = \text{const.}$ was constructed by means of linear interpolation. It must be noted that formula (143) yields for $\xi = 0$ and hence $\sigma = 0$ only the singular stream line (one point) $\psi = 0$ with $y^* = 0$ (cf. (85)). Since, however, for $\xi = 0$, u/U_0 must equal 1, one has because of the second equation (139)

$$\psi(0, y^*) = y^* \quad (145)$$

This result is naturally yielded also by our system of formulas: according to (85), namely,

$$\frac{y^*}{\sigma} = 2 \int_0^{\tau} \frac{\tau d\tau}{U(\tau, \sigma)} \quad (146)$$

whereas from (143) follows

$$\frac{\psi}{\sigma} = \frac{1}{2} (\tau^2 + \sigma) \quad (147)$$

The right side of (146) goes, due to the monotone growth of $U(\tau, \sigma)$ as function of τ from 0 to 2, with increasing τ from higher values toward

$$2 \int_0^{\pi} \frac{\tau}{2} d\tau = \frac{\tau^2}{2}$$

so that for $\sigma \rightarrow 0$ and large τ (and only these have to be considered for $\sigma \rightarrow 0$, according to (143),) the right sides of (146) and (147) agree. That, however, leads to (145).

Fig. 12 presents the stream line pattern, fig. 13 a partial section for ξ -values from 0 to 1 in magnified scale. Particularly noteworthy is the steepening of the stream lines at the beginning of the plate $\xi = 0$ where all stream lines have a vertical tangent (cf. also paragraph 16). All stream lines approach the wall vertically, since there $v = v_0$ and $u = 0$.

16. The Velocity Component Perpendicular to the Wall

The velocity component

$$v(\xi, y^*) / -v_0$$

perpendicular to the wall results from (139). One calculates

$$\frac{\partial \psi}{\partial \xi} = \frac{\partial \psi}{\partial \sigma} \frac{\partial \sigma}{\partial \xi} + \frac{\partial \psi}{\partial \tau} \frac{\partial \tau}{\partial \xi} \quad (148)$$

Now, according to (85),

$$d\xi = \sigma d\sigma, \quad dy^* = 2\sigma \frac{\tau}{U(\tau, \sigma)} d\tau + \frac{\partial y^*}{\partial \sigma} d\sigma \quad (149)$$

with

$$\frac{\partial y^*}{\partial \sigma} = \frac{y^*}{\sigma} - 2\sigma \int_0^{\tau} \frac{\tau \frac{\partial U(\tau, \sigma)}{\partial \sigma}}{U^2(\tau, \sigma)} d\tau \quad (150)$$

By further calculation one obtains from (149)

$$d\sigma = \frac{d\xi}{\sigma}, \quad d\tau = \frac{U(\tau, \sigma)}{2\sigma^2} \left(dy^* - \frac{dy^*}{d\sigma} \frac{d\xi}{\sigma} \right) \quad (151)$$

Since from (143) follows

$$\frac{\partial \Psi}{\partial \sigma} = \frac{1}{2} (\tau^2 + 2\sigma), \quad \frac{\partial \Psi}{\partial \tau} = \sigma\tau \quad (152)$$

one obtains, according to (148),

$$-\frac{v}{v_0} = \frac{\partial \Psi}{\partial \xi} = \frac{1}{2\sigma} (\tau^2 + 2\sigma) - \frac{U(\tau, \sigma)}{2\sigma} \frac{\partial y^*}{\partial \sigma} \quad (153)$$

This is the formula for calculation of the velocity component perpendicular to the wall. $\partial y^*/\partial \sigma$ can be determined by interpolation, since $y^*(\tau, \sigma)$ was required anyway for the former calculations.

For $\tau = 0$, where

$$U(0, \sigma) = 0 \quad \text{and} \quad \partial y^*(0, \sigma)/\partial \sigma = 0$$

necessarily follows from (153) $v/-v_0 = -1$, that is $v = v_0$. For $\tau \neq 0$ and $\sigma \rightarrow 0$ the decisive factor is, according to (153) and (150),

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \left(\frac{1}{2} \tau^2 - \frac{U}{\sigma} \frac{y^*}{\sigma} \right) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \left(\frac{1}{2} \tau^2 - U \int_0^{\tau} \frac{\tau d\tau}{U(\tau, \sigma)} \right) \quad (154)$$

equation (85) was taken into consideration. Due to the monotone growth of $U(\tau, \sigma)$ from 0 to 2 with τ the bracket is negative so that one obtains

$$\frac{v}{-v_0} = +\infty \quad \text{for} \quad \tau = 0 \quad \text{and} \quad \sigma = 0 \quad (155)$$

This, however, means nothing else but the steepening of the stream lines for $\xi = 0$ pointed out at the end of the preceding paragraph. Moreover,

one may conclude from (154) in connection with the two equations (85), that for $\xi \rightarrow 0$ and y^* different from zero

$$\frac{v}{-v_0} = \frac{1}{\sqrt{2\xi}} \lim_{\tau \rightarrow \infty} \left[U(\tau, 0) \int_0^\tau \frac{\tau d\tau}{U(\tau, 0)} - \frac{1}{2} \tau^2 \right] + \dots$$

Instead of calculating $\lim_{\tau \rightarrow \infty}$ it is sufficient to substitute that value τ^* for which $U(\tau^*, 0) = 2$ is satisfied with adequate accuracy. According to the calculations described in paragraph 11 one can put $\tau^* = 3$. Then one obtains

$$\begin{aligned} \frac{v}{-v_0} &= \frac{1}{\sqrt{2\xi}} \left[\int_0^3 \frac{\tau d\tau}{\frac{U(\tau, 0)}{2}} - 4.5 \right] + \dots \\ &= \frac{1}{\sqrt{2\xi}} \left[y_{B_1}^*(3) \frac{1}{\sqrt{2}} - 4.5 \right] + \dots \end{aligned}$$

$y_{B_1}^*$ being the value defined by (105). The numerical calculation gives the result

$$\frac{v}{-v_0} = \frac{0.8448}{\sqrt{\xi}} + \dots \quad (156)$$

For performing the calculation of the $\partial y^*/\partial \sigma$ for $\tau > 3$ appearing in (153) one should further note that, according to (85),

$$y^*(\tau) = y^*(3) + \frac{\sigma}{2} (\tau^2 - 9) \quad (\tau \geq 3) \quad (157)$$

Hence follows immediately

$$\frac{\partial y^*(\tau)}{\partial \sigma} = \frac{\partial y^*(3)}{\partial \sigma} + \frac{1}{2} (\tau^2 - 9) \quad (\tau \geq 3) \quad (158)$$

and thus from (153) that

$$\frac{v(\tau)}{-v_0} = \frac{v(3)}{-v_0} \quad \text{for } \tau \geq 3 \quad (159)$$

In fig. 14 the values of v/v_0 are plotted as function of y^* for the various values of ξ .

17. The Drag of the Plate

The total friction drag for the plate of the width b and the length l wetted on one side is

$$W = b \int_0^l \tau_0(x) dx \quad (160)$$

If one defines in the customary manner the coefficient of the friction drag by

$$c_f = \frac{W}{\frac{\rho}{2} U_0^2 l b} \quad (161)$$

and puts

$$f(\xi) = \frac{\tau_0}{\tau_{0\infty}} = \frac{\frac{\tau_0 \delta^*}{\mu U_0}}{-\frac{v_0 \delta^*}{v}} = \frac{\tau_0}{\rho(-v_0) U_0} \quad (162)$$

(regarding the value $\tau_{0\infty}$ for the "asymptotic solution" (reference 6)
 (also see paragraph 18) c_F becomes with

$$\xi_1 = \left(-\frac{v_0}{U_0}\right)^2 \frac{U_0 l}{\nu} = c_Q^2 \frac{U_0 l}{\nu} \quad (163)$$

$$c_F = 2 \frac{-v_0}{U_0} \frac{1}{\xi_1} \int_0^{\xi_1} \frac{\tau_0}{\tau_{0\infty}} d\xi = 2 \frac{-v_0}{U_0} F(\xi_1) = 2c_Q F(\xi_1) \quad (164)$$

with

$$F(\xi_1) = \frac{1}{\xi_1} \int_0^{\xi_1} f(\xi) d\xi \quad (165)$$

One concludes immediately:

For $\xi_1 \rightarrow \infty$ one obtains

$$F(\xi_1) \rightarrow 1, \quad c_F \rightarrow 2 \frac{-v_0}{U_0} = c_{F\infty} = 2c_Q$$

for $\xi_1 \rightarrow 0$ one obtains

$$F(\xi_1) \rightarrow \frac{0.6641}{\sqrt{\xi_1}}; \quad c_F \rightarrow \frac{-v_0}{U_0} \frac{1.3282}{\sqrt{\xi_1}} = \frac{1.3282}{\sqrt{\frac{U_0 l}{\nu}}}$$

The first is the drag law of the asymptotic solution, the latter Blasius' drag law for the case without suction. The drag law in the entrance region is given by equations (163) and (164). The drag coefficients for various mass coefficients c_Q coincide in one curve when

$$c_f/c_{f\infty} = c_f/2c_{Q_0} = F(\xi_2)$$

is plotted against

$$c_{Q_0}^2 U_0^2 / \nu = \xi_2$$

(Fig. 15).

For calculation of the integral (165) one introduces according to (85) σ as integration variable:

$$F(\xi_2) = \frac{2}{2\sigma_2^2} \int_0^{\sigma_2} f_1(\sigma) \sigma d\sigma \quad (166)$$

with

$$\frac{1}{2} \sigma_2^2 = \xi_1 \quad \text{and} \quad f_1(\sigma) = f(\xi) \quad (167)$$

According to (87) and (92) as well as to (106), (109) and (110)

$$-\frac{\delta^* v_0}{\nu} = \frac{1}{\sqrt{2}} 1.7207\sigma \dots, \quad \frac{\tau_0 \delta^*}{\mu U_0} = 0.5714 + \dots$$

hence

$$\begin{aligned} F(\xi_2) &= \frac{2}{\sigma_2^2} \int_0^{\sigma_2} \sqrt{2} \frac{0.5714}{1.7207} d\sigma + \dots = \frac{2\sqrt{2}}{\sigma_2} \frac{0.5714}{1.7207} + \dots \\ &= 0.6641 \frac{1}{\sqrt{\xi_1}} \end{aligned} \quad (168)$$

in agreement with the statement made in connection with (165). The integral appearing in (166) is a definite integral and was evaluated according to Simpson's rule; first with the width of interval $h = 0.2$; thus one obtained the values of $F(\xi_1)$ for $\sigma_1 = 0, 0.2, 0.4, \dots$. From them the value of F for $\sigma_1 = 0.1$ was determined according to Newton's interpolation formula; by repeated successive application of Simpson's rule the F values for $\sigma_1 = 0.3, 0.5, 0.7, \dots$ were obtained.

The last column of table 2 gives the values of the function $F(\xi_1)$. Fig. 15 shows a sketch of this function. In fig. 16 the curves $10^3 c_f$ are plotted as function of $U_0 l / \nu$ with $c_Q = -v_0 / U_0$ as parameter; for comparison the curves are drawn as dashed lines which illustrate for the flow without suction the transition from laminar to turbulent flow or, respectively, the fully turbulent flow (reference 17).

The difference between the fully turbulent drag curve and a curve $c_Q = \text{const.}$ gives the drag reduction by means of keeping the boundary layer laminar, under the presupposition that the laminar boundary layer is stable for the respective mass coefficient. that is, that no transition to the turbulent flow type occurs⁴.

18. The Asymptotic Solution

According to Schlichting (reference 6) the problem (2), (3) is solved by the expression

$$v = v_0, u = U_0(1 - e^{-y^*}) \quad (169)$$

⁴Information on the mass coefficients required for maintenance of a laminar boundary layer is given in an investigation by A. Ulrich (see reference 13) in which the laminar velocity profiles calculated in the present report were investigated for stability. The mass coefficient $c_{Q_{\text{crit}}} = 1.2 \times 10^{-4}$ was found to be sufficient for the maintenance of a laminar boundary layer. Hence reductions in friction drag of the order of magnitude of 70 to 80 percent of the fully turbulent friction drag result for the Reynolds number in the region

$$U_0 l / \nu = 5 \times 10^6 \text{ to } 10^8$$

important in practice.

which also satisfies the boundary conditions (4) with the exception of the fourth, the latter only for $y^* \rightarrow \infty$. With increasing y^* the individual velocity profiles u/U_0 will tend toward this "asymptotic solution" (169). We shall reconvert (169) to our calculation coordinates σ and τ :

$$1 - \frac{u}{U_0} = e^{-y^*}, \quad y^* = -\ln \left(1 - \frac{u}{U_0} \right)$$

One takes (85) and (132) into consideration:

$$2\sigma \int_0^{\tau} \frac{\tau d\tau}{\sqrt{v}} = -\ln \left(1 - \frac{1}{2} \sqrt{v} \right) \quad (170)$$

Differentiation with respect to τ gives

$$\frac{2\sigma\tau}{\sqrt{v}} = \frac{\frac{1}{2} (\sqrt{v})'}{1 - \frac{1}{2} \sqrt{v}}, \quad -\frac{\frac{1}{2} \sqrt{v} d\sqrt{v}}{1 - \frac{1}{2} \sqrt{v}} = -2\sigma\tau d\tau$$

and integration

$$\sqrt{v} + 2\ln \left(1 - \frac{1}{2} \sqrt{v} \right) = -\sigma\tau^2 \quad (171)$$

The additive integration constant which actually should appear on the right must disappear since according to (170) for $\tau = 0$ \sqrt{v} also must disappear.

From (171) one concludes for instance: For constant τ \sqrt{v} tends with growing σ increasingly toward 2; this is the reason for the remark made in paragraph 12 that the quantity called τ_1 there decreases with growing σ .

Equation (171) assumes a clearer form if one introduces according to (33), (34) and (49) X and Y as independent variables:

$$\sqrt{R} + 2\ln\left(1 - \frac{1}{2}\sqrt{R}\right) = -X \quad (172)$$

This could have been confirmed directly from (47).

In all figures the values referred to the asymptotic solution which can easily be calculated are marked in. It is shown that for $\sigma = 4$ all flow characteristics lie sufficiently close to the corresponding values of the asymptotic solution.

SUMMARY

A general calculation method is given with which to determine the boundary layer developing along an infinitely extended flat plate under the influence of an arbitrary suction or blowing law. For the special case of homogeneous suction the numerical calculation of the velocity within the boundary layer is performed completely. Two numerical methods are applied which are both fundamentally based on the approximate solution of an ordinary linear differential equation of the second order for the calculation of each velocity profile vertical to the plate; thus one attains a sufficiently accurate determination of the first and second derivatives of the velocity vertical to the wall as well. Both methods can also be applied for the continuation of an arbitrarily prescribed velocity profile.

Translated by Mary L. Mahler,
National Advisory Committee
for Aeronautics

REFERENCES

1. Betz, A.: Schr. d. Dt. Akad. d. Luftfahrtforschg. H. 49; see also Jb. d. Dt. Akad. d. Luftfahrtforschg. 1939/40, p. 246.
2. Schlichting, H.: Vortrag vor d. Dt. Akad. d. Luftfahrtforschg., May 7, 1943, Jb. d. Dt. Akad. d. Luftfahrtforschg. 1943/44, p. 90.
3. Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhandlung III. Intern. Math. Kongress, Heidelberg 1904. Reprinted in Vier Adhdlg. zur Hydrodynamik, Göttingen 1927.
4. Blasius, H.: Zs. Math. u. Phys., Bd. 56 (1908), p. 1.
5. Schlichting, H. and Bussmann, K.: Schr. d. Dt. Akad. d. Luftfahrtforschg., Bd. 7 B (1943), p. 25.
6. Schlichting, H.: Luftfahrtforschg. Bd. 19 (1942), p. 179.
7. Lamb, H.: Textbook of Hydrodynamics, 2nd Edition, according to the 5th English edition edited by E. Helly, with preface and supplements by R. v. Mises, Leipzig and Berlin 1931. (See also R. v. Mises, Zs. f. angew. Math. u. Mech., Bd. 18 (1938), p. 77.
8. Prandtl, L.: Zs. angew. Math. u. Mech., Bd. 18 (1938), p. 77.
9. Görtler, H.: Zs. angew. Math. u. Mech., Bd. 19 (1939), p. 129.
10. Schröder, K.: Ein einfaches numerisches Verfahren zur Berechnung der laminaren Grenzschicht. FB 1741 (1943).
11. Tollmien, W.: Nachr. Ges. Wiss. Göttingen. Math. Phys. Klasse (1929), p. 21.
12. Schlichting, H.: Jb. 1940 d. dt. Luftfahrtforschg., p. 97.
13. Ulrich, A.: Ber. 44/8d. Aerodynam. Inst. d. Techn. Hochschule Braunschweig. (Soon to be published in the literature of the Dt. Akad. d. Luftfahrtforschg.)
14. Schulz, G.: Formelsammlung zur praktischen Mathematik. Sammlung Göschen Bd. 1110.
15. Howarth, L.: Proc. Roy. Soc. London, Ser. A, No. 919, Bd. 164 (1938), p. 547.

16. Luckert, H. J.: Schr. d. Mathemat. Seminars u. d. Inst. f. angew. Mathematik d. Univ. Berlin, Bd. 1 (1933), H. 8.
17. Prandtl, L.: Ergebnisse der Aerodynamischen Versuchsanstalt Göttingen, IV Lfrg., 1932, p. 18. Or Prandtl, L.: Führer durch die Strömungslehre, Braunschweig 1942, p. 175.

TABLE I

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 0; \xi = 0$ Blasius				$\sigma = 0.1; \xi = 0.005$			
$\eta^* = y\sqrt{\frac{U_0}{\nu x}}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial \eta^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial \eta^{*2}}$	$y^* = \frac{-v_0 y}{\nu}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	0.332	0	0	0	5.322	-5.329
0.2064	0.0685	0.332	-0.001	0.0137	0.0726	5.248	-5.481
0.4128	0.1370	0.331	-0.005	0.0275	0.1445	5.169	-6.100
0.6193	0.2053	0.330	-0.011	0.0414	0.2156	5.077	-7.149
0.7935	0.2733	0.327	-0.018	0.0554	0.2858	4.968	-8.595
1.0007	0.3406	0.322	-0.028	0.0694	0.3546	4.835	-10.389
1.2088	0.4069	0.315	-0.040	0.0836	0.4220	4.673	-12.459
1.4179	0.4718	0.305	-0.053	0.0979	0.4874	4.479	-14.705
1.6286	0.5348	0.293	-0.066	0.1123	0.5505	4.250	-17.002
1.8412	0.5955	0.277	-0.079	0.1269	0.6108	3.985	-19.204
2.0563	0.6532	0.259	-0.092	0.1418	0.6678	3.685	-21.150
2.2744	0.7074	0.238	-0.102	0.1569	0.7210	3.353	-22.676
2.4964	0.7575	0.214	-0.109	0.1723	0.7699	2.995	-23.626
2.7227	0.8032	0.189	-0.113	0.1881	0.8142	2.620	-23.880
2.9544	0.8440	0.163	-0.113	0.2043	0.8535	2.237	-23.361
3.1922	0.8795	0.136	-0.109	0.2209	0.8875	1.858	-22.062
3.4371	0.9097	0.111	-0.100	0.2381	0.9163	1.495	-20.051
3.6900	0.9347	0.087	-0.089	0.2558	0.9398	1.161	-17.472
3.9519	0.9545	0.065	-0.075	0.2743	0.9584	0.866	-14.537
4.2237	0.9697	0.047	-0.060	0.2934	0.9725	0.617	-11.491
4.5063	0.9808	0.032	-0.045	0.3134	0.9827	0.425	-8.586
4.8007	0.9885	0.021	-0.032	0.3341	0.9898	0.267	-6.027
5.1074	0.9935	0.013	-0.022	0.3558	0.9943	0.160	-3.949
5.4271	0.9966	0.007	-0.013	0.3784	0.9970	0.089	-2.399
5.7603	0.9984	0.004	-0.008	0.4020	0.9986	0.046	-1.342
6.1072	0.9993	0.002	-0.004	0.4265	0.9994	0.022	-0.690
6.4680	0.9997	0.001	-0.002	0.4520	0.9998	0.009	-0.323
6.8428	0.9999	0	-0.001	0.4785	0.9999	0.004	-0.136
7.2317	1.0000	0	0	0.5060	1	0.001	-0.051
7.6348	1.0000	0	0	0.5344	1	0	-0.012
8.0519	1.0000	0	0	0.5640	1	0	-0.006

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 0.2; \xi = 0.02$				$\sigma = 0.3; \xi = 0.045$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	2.986	-2.96	0	0	2.216	-2.20
0.0260	0.0766	2.910	-2.94	0.0370	0.0805	2.136	-2.15
0.0522	0.1519	2.831	-3.04	0.0745	0.1591	2.055	-2.14
0.0787	0.2257	2.748	-3.24	0.1125	0.2356	1.973	-2.19
0.1054	0.2980	2.658	-3.53	0.1509	0.3098	1.888	-2.27
0.1324	0.3684	2.558	-3.90	0.1899	0.3817	1.797	-2.38
0.1597	0.4310	2.445	-4.27	0.2295	0.4510	1.701	-2.49
0.1874	0.5027	2.319	-4.79	0.2698	0.5174	1.597	-2.67
0.2154	0.5658	2.178	-5.25	0.3107	0.5805	1.485	-2.79
0.2439	0.6257	2.023	-5.67	0.3525	0.6400	1.365	-2.91
0.2730	0.6820	1.852	-6.02	0.3951	0.6956	1.239	-3.00
0.3026	0.7342	1.670	-6.26	0.4388	0.7468	1.107	-3.04
0.3329	0.7820	1.478	-6.35	0.4837	0.7933	0.972	-3.01
0.3640	0.8249	1.282	-6.28	0.5297	0.8349	0.835	-2.92
0.3960	0.8627	1.084	-6.02	0.5772	0.8713	0.700	-2.75
0.4290	0.8952	0.893	-5.59	0.6262	0.9024	0.571	-2.51
0.4631	0.9225	0.712	-5.00	0.6770	0.9283	0.451	-2.21
0.4984	0.9442	0.548	-4.29	0.7297	0.9492	0.344	-1.87
0.5351	0.9621	0.405	-3.52	0.7845	0.9654	0.252	-1.51
0.5733	0.9751	0.285	-2.74	0.8416	0.9775	0.176	-1.16
0.6131	0.9845	0.191	-2.02	0.9011	0.9861	0.116	-0.84
0.6546	0.9909	0.121	-1.40	0.9633	0.9919	0.073	-0.58
0.6979	0.9950	0.071	-0.90	1.0282	0.9956	0.043	-0.367
0.7431	0.9974	0.039	-0.54	1.0959	0.9978	0.023	-0.217
0.7902	0.9988	0.020	-0.299	1.1665	0.9990	0.012	-0.119
0.8392	0.9995	0.009	-0.150	1.2401	0.9995	0.005	-0.059
0.8902	0.9998	0.004	-0.068	1.3166	0.9998	0.002	-0.027
0.9432	0.9999	0.001	-0.027	1.3961	0.9999	0.001	-0.011
0.9982	1	0.001	-0.011	1.4786	1	0	-0.004
1.0552	1	0	-0.004	1.5641	1	0	-0.001
1.1142	1	0	-0.001	1.6526	1	0	-0.001

TABLE I.-- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 0.4; \xi = 0.08$				$\sigma = 0.5; \xi = 0.125$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.835	-1.82	0	0	1.612	-1.60
0.0471	0.0843	1.751	-1.75	0.0562	0.0882	1.525	-1.52
0.0948	0.1661	1.669	-1.72	0.1135	0.1730	1.439	-1.47
0.1434	0.2452	1.586	-1.70	0.1718	0.2545	1.355	-1.43
0.1928	0.3214	1.502	-1.71	0.2314	0.3327	1.271	-1.40
0.2430	0.3947	1.415	-1.73	0.2921	0.4073	1.187	-1.38
0.2941	0.4648	1.326	-1.77	0.3542	0.4783	1.101	-1.37
0.3463	0.5315	1.233	-1.80	0.4176	0.5454	1.014	-1.37
0.3995	0.5946	1.136	-1.84	0.4826	0.6084	0.926	-1.36
0.4539	0.6537	1.035	-1.86	0.5492	0.6671	0.836	-1.34
0.5097	0.7085	0.931	-1.87	0.6176	0.7211	0.745	-1.31
0.5669	0.7587	0.825	-1.85	0.6880	0.7704	0.654	-1.27
0.6257	0.8041	0.717	-1.80	0.7605	0.8145	0.564	-1.21
0.6864	0.8443	0.611	-1.71	0.8354	0.8534	0.476	-1.13
0.7490	0.8793	0.508	-1.58	0.9129	0.8871	0.393	-1.03
0.8138	0.9090	0.411	-1.42	0.9933	0.9154	0.315	-0.91
0.8811	0.9336	0.322	-1.23	1.0768	0.9387	0.244	-0.78
0.9510	0.9533	0.243	-1.02	1.1638	0.9572	0.183	-0.64
1.0238	0.9684	0.176	-0.82	1.2545	0.9713	0.131	-0.50
1.0998	0.9797	0.122	-0.62	1.3493	0.9816	0.090	-0.376
1.1790	0.9876	0.080	-0.444	1.4482	0.9889	0.058	-0.266
1.2618	0.9928	0.049	-0.299	1.5516	0.9937	0.036	-0.177
1.3483	0.9961	0.029	-0.186	1.6596	0.9966	0.020	-0.110
1.4385	0.9981	0.015	-0.110	1.7724	0.9983	0.011	-0.063
1.5327	0.9991	0.008	-0.059	1.8900	0.9992	0.005	-0.033
1.6307	0.9996	0.004	-0.029	2.0126	0.9997	0.002	-0.016
1.7328	0.9999	0.001	-0.012	2.1401	0.9999	0.001	-0.007
1.8388	1.0000	0.001	-0.005	2.2726	1.0000	0	-0.003
1.9488	1	0	-0.001	2.4101	1	0	-0.001
2.0628	1	0	0	2.5526	1	0	-0.001
2.1808	1	0	0	2.7001	1	0	0

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 0.6; \xi = 0.18$				$\sigma = 0.7; \xi = 0.245$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.467	-1.45	0	0	1.366	-1.35
0.0647	0.0919	1.376	-1.37	0.0724	0.0955	1.271	-1.27
0.1307	0.1798	1.287	-1.30	0.1464	0.1864	1.180	-1.19
0.1982	0.2637	1.201	-1.25	0.2225	0.2727	1.092	-1.12
0.2672	0.3437	1.117	-1.20	0.3005	0.3545	1.007	-1.07
0.3379	0.4197	1.033	-1.16	0.3805	0.4317	0.923	-1.02
0.4102	0.4949	0.950	-1.13	0.4627	0.5042	0.842	-0.97
0.4844	0.5593	0.868	-1.101	0.5471	0.5719	0.762	-0.93
0.5606	0.6218	0.785	-1.07	0.6341	0.6347	0.683	-0.88
0.6388	0.6800	0.703	-1.04	0.7237	0.6924	0.606	-0.84
0.7194	0.7333	0.621	-0.99	0.8161	0.7450	0.531	-0.79
0.8026	0.7815	0.540	-0.94	0.9117	0.7922	0.458	-0.74
0.8884	0.8245	0.462	-0.88	1.0106	0.8340	0.388	-0.68
0.9773	0.8621	0.386	-0.81	1.1132	0.8704	0.322	-0.61
1.0695	0.8944	0.316	-0.73	1.2206	0.9013	0.261	-0.54
1.1653	0.9215	0.251	-0.63	1.3316	0.9270	0.205	-0.462
1.2649	0.9435	0.193	-0.532	1.4473	0.9479	0.156	-0.384
1.3689	0.9608	0.143	-0.413	1.5680	0.9642	0.115	-0.307
1.4774	0.9735	0.101	-0.334	1.6942	0.9764	0.081	-0.235
1.5907	0.9835	0.069	-0.247	1.8262	0.9852	0.054	-0.171
1.7093	0.9901	0.044	-0.172	1.9643	0.9907	0.034	-0.117
1.8332	0.9944	0.027	-0.113	2.1088	0.9951	0.020	-0.076
1.9628	0.9970	0.015	-0.069	2.2599	0.9974	0.011	-0.046
2.0981	0.9985	0.008	-0.039	2.4177	0.9987	0.006	-0.026
2.2392	0.9993	0.004	-0.020	2.5823	0.9994	0.003	-0.013
2.3863	0.9997	0.002	-0.010	2.7539	0.9998	0.001	-0.006
2.5393	0.9999	0.001	-0.004	2.9324	0.9999	0.001	-0.003
2.6983	0.9999	0	-0.002	3.1179	1	0	-0.001
2.8633	1	0	-0.001	3.3104	1	0	-0.001
3.0343	1	0	0	3.5099	1	0	0
3.2113	1	0	0	3.7164	1	0	0

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND
SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 0.8; \xi = 0.32$				$\sigma = 0.9; \xi = 0.405$			
$y^* = \frac{-v_o y}{v}$	$\frac{u}{U_o}$	$\frac{1}{U_o} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_o} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_o y}{v}$	$\frac{u}{U_o}$	$\frac{1}{U_o} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_o} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.292	-1.28	0	0	1.237	-1.23
0.0797	0.0990	1.194	-1.19	0.0865	0.1025	1.135	-1.13
0.1616	0.1929	1.100	-1.11	0.1756	0.1993	1.038	-1.04
0.2457	0.2815	1.010	-1.03	0.2673	0.2902	0.946	-0.96
0.3321	0.3651	0.923	-0.97	0.3618	0.3754	0.859	-0.89
0.4211	0.4434	0.840	-0.91	0.4592	0.4549	0.775	-0.83
0.5126	0.5166	0.760	-0.85	0.5597	0.5287	0.695	-0.76
0.6070	0.5846	0.682	-0.80	0.6635	0.5968	0.619	-0.71
0.7043	0.6472	0.606	-0.75	0.7709	0.6593	0.546	-0.65
0.8048	0.7044	0.533	-0.70	0.8820	0.7160	0.476	-0.60
0.9088	0.7562	0.463	-0.65	0.9972	0.7670	0.410	-0.55
1.0165	0.8024	0.396	-0.60	1.1168	0.8122	0.347	-0.495
1.1283	0.8431	0.333	-0.54	1.2411	0.8517	0.289	-0.441
1.2444	0.8782	0.274	-0.479	1.3706	0.8856	0.236	-0.386
1.3652	0.9079	0.219	-0.416	1.5055	0.9141	0.187	-0.331
1.4912	0.9324	0.171	-0.352	1.6464	0.9374	0.145	-0.276
1.6228	0.9521	0.129	-0.288	1.7937	0.9559	0.108	-0.222
1.7603	0.9673	0.094	-0.227	1.9478	0.9701	0.078	-0.173
1.9041	0.9786	0.065	-0.171	2.1093	0.9807	0.054	-0.129
2.0547	0.9867	0.043	-0.123	2.2784	0.9881	0.035	-0.091
2.2123	0.9922	0.027	-0.083	2.4555	0.9930	0.022	-0.061
2.3773	0.9957	0.016	-0.053	2.6410	0.9962	0.013	-0.038
2.5498	0.9977	0.009	-0.032	2.8350	0.9980	0.007	-0.023
2.7301	0.9989	0.005	-0.018	3.0378	0.9991	0.004	-0.012
2.9183	0.9995	0.002	-0.009	3.2494	0.9996	0.002	-0.006
3.1143	0.9998	0.001	-0.004	3.4700	0.9998	0.001	-0.003
3.3184	0.9999	0	-0.002	3.6995	0.9999	0	-0.001
3.5304	1	0	-0.001	3.9380	1	0	0
3.7504	1	0	0	4.1855	1	0	0
3.9784	1	0	0	4.4420	1	0	0
4.2144	1	0	0	4.7075	1	0	0

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 1.0; \xi = 0.5$				$\sigma = 1.2; \xi = 0.72$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.194	-1.18	0	0	1.135	-1.135
0.0929	0.1060	1.089	-1.08	0.1047	0.1127	1.022	-1.022
0.1888	0.2055	0.989	-0.99	0.2130	0.2176	0.917	-0.921
0.2877	0.2986	0.895	-0.91	0.3253	0.3149	0.818	-0.828
0.3898	0.3854	0.806	-0.83	0.4417	0.4048	0.727	-0.744
0.4953	0.4660	0.722	-0.76	0.5625	0.4874	0.642	-0.665
0.6044	0.5405	0.643	-0.70	0.6880	0.5629	0.563	-0.596
0.7174	0.6088	0.568	-0.64	0.8184	0.6315	0.489	-0.532
0.8345	0.6710	0.497	-0.58	0.9541	0.6932	0.422	-0.477
0.9560	0.7272	0.430	-0.52	1.0955	0.7483	0.359	-0.424
1.0821	0.7773	0.367	-0.472	1.2430	0.7968	0.301	-0.365
1.2134	0.8215	0.308	-0.420	1.3969	0.8391	0.249	-0.316
1.3501	0.8599	0.254	-0.369	1.5578	0.8753	0.202	-0.262
1.4926	0.8976	0.206	-0.318	1.7261	0.9056	0.160	-0.228
1.6415	0.9199	0.162	-0.268	1.9025	0.9306	0.136	-0.194
1.7971	0.9421	0.124	-0.221	2.0874	0.9505	0.093	-0.154
1.9601	0.9595	0.092	-0.176	2.2814	0.9660	0.067	-0.120
2.1308	0.9778	0.065	-0.135	2.4851	0.9775	0.047	-0.086
2.3097	0.9825	0.045	-0.099	2.6989	0.9858	0.031	-0.062
2.4973	0.9893	0.029	-0.069	2.9234	0.9915	0.020	-0.043
2.6940	0.9938	0.018	-0.046	3.1590	0.9952	0.012	-0.026
2.8999	0.9967	0.010	-0.028	3.4059	0.9974	0.007	-0.015
3.1154	0.9983	0.005	-0.016	3.6643	0.9987	0.004	-0.008
3.3407	0.9992	0.003	-0.009	3.9346	0.9994	0.002	-0.005
3.5758	0.9996	0.001	-0.004	4.2167	0.9998	0.001	0
3.8209	0.9999	0.001	-0.002	4.5107	0.9999	0	0
4.0759	0.9999	0	-0.001	4.8168	1	0	0
4.3409	1	0	0	5.1348	1	0	0
4.6159	1	0	0	5.4648	1	0	0
4.9009	1	0	0				
5.1959	1	0	0				

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 1.4; \xi = 0.98$				$\sigma = 1.6; \xi = 1.28$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.094	-1.094	0	0	1.068	-1.068
0.1153	0.1192	0.975	-0.975	0.1250	0.1254	0.943	-0.944
0.2350	0.2293	0.865	-0.868	0.2552	0.2406	0.827	-0.830
0.3596	0.3306	0.763	-0.770	0.3912	0.3457	0.722	-0.726
0.4892	0.4234	0.670	-0.680	0.5331	0.4411	0.625	-0.633
0.6242	0.5079	0.583	-0.599	0.6815	0.5273	0.538	-0.549
0.7650	0.5843	0.504	-0.526	0.8367	0.6045	0.459	-0.473
0.9119	0.6544	0.432	-0.460	0.9993	0.6731	0.387	-0.406
1.0652	0.7140	0.366	-0.399	1.1697	0.7335	0.324	-0.345
1.2257	0.7679	0.307	-0.344	1.3485	0.7862	0.267	-0.291
1.3936	0.8148	0.253	-0.294	1.5362	0.8314	0.217	-0.243
1.5695	0.8551	0.206	-0.248	1.7336	0.8698	0.173	-0.200
1.7540	0.8891	0.164	-0.207	1.9412	0.9017	0.136	-0.163
1.9476	0.9173	0.128	-0.169	2.1597	0.9277	0.104	-0.130
2.1510	0.9400	0.097	-0.135	2.3898	0.9484	0.077	-0.101
2.3648	0.9579	0.071	-0.105	2.6323	0.9643	0.056	-0.077
2.5897	0.9715	0.051	-0.079	2.8878	0.9763	0.039	-0.056
2.8261	0.9815	0.035	-0.058	3.1570	0.9849	0.026	-0.040
3.0748	0.9886	0.023	-0.040	3.4403	0.9908	0.017	-0.027
3.3361	0.9933	0.014	-0.027	3.7384	0.9947	0.010	-0.017
3.6105	0.9963	0.008	-0.016	4.0517	0.9971	0.006	-0.011
3.8983	0.9981	0.005	-0.010	4.3804	0.9984	0.003	-0.006
4.1998	0.9991	0.002	-0.005	4.7248	0.9993	0.002	-0.003
4.5150	0.9996	0.001	-0.003	5.0849	0.9997	0.001	-0.002
4.8441	0.9998	0.000	-0.001	5.4610	0.9999	0	-0.001
5.1871	0.9999	0	0	5.8530	1	0	0
5.5441	1	0	0	6.2610	1	0	0
5.9151	1	0	0	6.6850	1	0	0

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND
SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 2.0; \xi = 2.0$				$\sigma = 2.4; \xi = 2.88$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.036	-1.037	0	0	1.019	-1.020
0.1422	0.1374	0.899	-0.900	0.1575	0.1485	0.871	-0.871
0.2914	0.2619	0.774	-0.775	0.3235	0.2817	0.738	-0.738
0.4479	0.3740	0.662	-0.664	0.4986	0.4001	0.619	-0.620
0.6125	0.4743	0.561	-0.565	0.6836	0.5047	0.514	-0.516
0.7855	0.5633	0.471	-0.476	0.8793	0.5960	0.422	-0.425
0.9677	0.6417	0.391	-0.398	1.0866	0.6750	0.343	-0.346
1.1598	0.7100	0.322	-0.330	1.3063	0.7425	0.274	-0.279
1.3624	0.7688	0.261	-0.271	1.5394	0.7994	0.216	-0.221
1.5763	0.8188	0.209	-0.220	1.7870	0.8467	0.168	-0.173
1.8024	0.8607	0.164	-0.176	2.0500	0.8853	0.128	-0.133
2.0414	0.8952	0.126	-0.139	2.3295	0.9162	0.095	-0.101
2.2942	0.9231	0.095	-0.108	2.6267	0.9404	0.069	-0.075
2.5617	0.9451	0.070	-0.082	2.9424	0.9588	0.049	-0.054
2.8447	0.9621	0.050	-0.061	3.2778	0.9725	0.034	-0.038
3.1440	0.9747	0.035	-0.044	3.6337	0.9823	0.022	-0.026
3.4605	0.9838	0.023	-0.030	4.0110	0.9891	0.014	-0.017
3.7949	0.9900	0.015	-0.020	4.4104	0.9935	0.009	-0.011
4.1476	0.9942	0.009	-0.013	4.8325	0.9964	0.005	-0.007
4.5193	0.9968	0.005	-0.008	5.2777	0.9981	0.003	-0.004
4.9102	0.9983	0.003	-0.005	5.7464	0.9990	0.001	-0.002
5.3207	0.9992	0.001	-0.002	6.2387	0.9996	0.001	-0.001
5.7510	0.9996	0.001	-0.001	6.7549	0.9998	0	-0.001
6.2011	0.9999	0	-0.001	7.2949	0.9999	0	0
6.6712	1	0	0	7.8589	1	0	0
7.1612	1	0	0	8.4469	1	0	0
7.6713	1	0	0	9.0589	1	0	0
8.2013	1	0	0	9.6949	1	0	0

TABLE I.- Continued

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND

SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = 3.2; \xi = 5.12$				$\sigma = 4.0; \xi = 8.0$			
$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$	$y^* = \frac{-v_0 y}{v}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1.009	-1.014	0	0	1.000	-0.996
0.1836	0.1691	0.839	-0.843	0.2068	0.1870	0.814	-0.811
0.3788	0.3178	0.690	-0.693	0.4284	0.3488	0.653	-0.651
0.5868	0.4471	0.560	-0.562	0.6661	0.4869	0.515	-0.514
0.8086	0.5584	0.448	-0.450	0.9218	0.6031	0.399	-0.399
1.0456	0.6529	0.353	-0.355	1.1973	0.6991	0.303	-0.303
1.2992	0.7319	0.273	-0.275	1.4947	0.7769	0.225	-0.225
1.5708	0.7969	0.208	-0.210	1.8159	0.8386	0.163	-0.163
1.8619	0.8494	0.155	-0.157	2.1632	0.8863	0.115	-0.115
2.1741	0.8909	0.113	-0.115	2.5387	0.9222	0.079	-0.079
2.5090	0.9229	0.080	-0.082	2.9446	0.9484	0.052	-0.053
2.8681	0.9471	0.056	-0.057	3.3828	0.9669	0.034	-0.034
3.2528	0.9648	0.037	-0.039	3.8552	0.9795	0.021	-0.021
3.6645	0.9773	0.024	-0.026	4.3633	0.9878	0.012	-0.013
4.1045	0.9859	0.015	-0.016	4.9085	0.9929	0.007	-0.007
4.5736	0.9916	0.009	-0.010	5.4916	0.9961	0.004	-0.004
5.0729	0.9952	0.005	-0.006	6.1134	0.9979	0.002	-0.002
5.6028	0.9973	0.003	-0.003	6.7744	0.9989	0.001	-0.001
6.1639	0.9986	0.002	-0.002	7.4750	0.9994	0.001	0
6.7565	0.9993	0.001	-0.001	8.2153	0.9997	0	0
7.3808	0.9997	0	0	8.9955	0.9998	0	0
8.0370	0.9998	0	0	9.8155	0.9999	0	0
8.7251	0.9999	0	0	10.6756	1	0	0
9.4451	1	0	0	11.5756	1	0	0
10.1971	1	0	0	12.5156	1	0	0
10.9811	1	0	0	13.4955	1	0	0
11.7971	1	0	0				

TABLE I.- Concluded

THE VELOCITY DISTRIBUTION WITH ITS FIRST AND
SECOND DERIVATIVE FOR VARIOUS ξ

$\sigma = \infty;$		$\xi = \infty$	
Asymptotic Suction Profile			
$y^* = \frac{-v_0 y}{\nu}$	$\frac{u}{U_0}$	$\frac{1}{U_0} \frac{\partial u}{\partial y^*}$	$\frac{1}{U_0} \frac{\partial^2 u}{\partial y^{*2}}$
0	0	1	-1
0.2	0.1813	0.8187	-0.8187
0.4	0.3297	0.6703	-0.6703
0.6	0.4512	0.5488	-0.5488
0.8	0.5507	0.4493	-0.4493
1.0	0.6321	0.3679	-0.3679
1.2	0.6989	0.3012	-0.3012
1.4	0.7534	0.2466	-0.2466
1.6	0.7981	0.2019	-0.2019
1.8	0.8347	0.1653	-0.1653
2.0	0.8647	0.1353	-0.1353
2.2	0.8892	0.1108	-0.1108
2.4	0.9093	0.0907	-0.0907
2.6	0.9257	0.0743	-0.0743
2.8	0.9392	0.0608	-0.0608
3.0	0.9502	0.0498	-0.0498
3.2	0.9592	0.0408	-0.0408
3.4	0.9666	0.0334	-0.0334
3.6	0.9727	0.0273	-0.0273
3.8	0.9776	0.0224	-0.0224
4.0	0.9817	0.0183	-0.0183
4.2	0.9850	0.0150	-0.0150
4.4	0.9877	0.0123	-0.0123
4.6	0.9899	0.0101	-0.0101
4.8	0.9918	0.0082	-0.0082
5.0	0.9933	0.0067	-0.0067
5.5	0.9959	0.0041	-0.0041
6.0	0.9972	0.0028	-0.0028
6.5	0.9985	0.0015	-0.0015
7.0	0.9991	0.0009	-0.0009

TABLE II

THE CHARACTERISTIC BOUNDARY-LAYER PARAMETERS:

$$-v_0\delta^*/\nu, -v_0\delta/\nu, \delta^*/\delta, \tau_0\delta^*/\mu U_0, F(\xi)$$

σ	ξ	$\sqrt{\xi}$	$\frac{-v_0\delta^*}{\nu}$	$\frac{-v_0\delta}{\nu}$	$\frac{\delta^*}{\delta}$	$\frac{\tau_0\delta^*}{\mu U_0}$	$F(\xi)$
	Blasius						
0	0	0	0	0	2.59	0.5714	∞
0.1	0.005	0.0707	0.1140	0.0451	2.53	0.6068	10.014
0.2	0.02	0.141	0.2114	0.0855	2.47	0.6311	5.327
0.3	0.045	0.212	0.3026	0.1246	2.43	0.6706	3.768
0.4	0.08	0.283	0.3808	0.1597	2.39	0.6988	2.994
0.5	0.125	0.354	0.4501	0.1917	2.35	0.7258	2.532
0.6	0.18	0.424	0.5113	0.2210	2.31	0.7500	2.227
0.7	0.245	0.495	0.5660	0.2480	2.28	0.7730	2.011
0.8	0.32	0.566	0.6143	0.2725	2.25	0.7937	1.850
0.9	0.405	0.636	0.6575	0.2949	2.23	0.8130	1.727
1.0	0.5	0.707	0.6954	0.3149	2.21	0.8302	1.629
1.2	0.72	0.849	0.7613	0.3505	2.17	0.8638	1.486
1.4	0.98	0.990	0.8121	0.3802	2.14	0.8888	1.387
1.6	1.28	1.131	0.8530	0.4044	2.11	0.9113	1.315
2.0	2.0	1.414	0.9108	0.4399	2.07	0.9436	1.220
2.4	2.88	1.697	0.9475	0.4630	2.05	0.9658	1.161
3.2	5.12	2.263	0.9828	0.4878	2.01	0.9914	1.0956
4.0	8.0	2.828	0.9958	0.4971	2.00	0.9961	1.063
∞	∞	∞	1	0.5	2	1	1

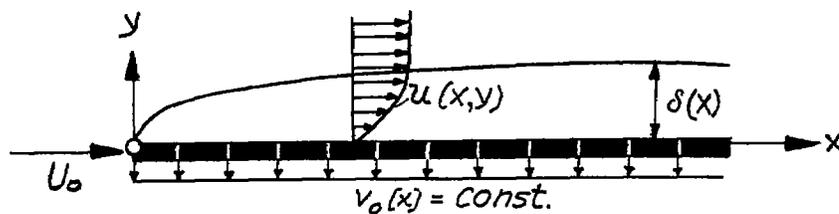


Figure 1.- Explanatory sketch; system of coordinates.

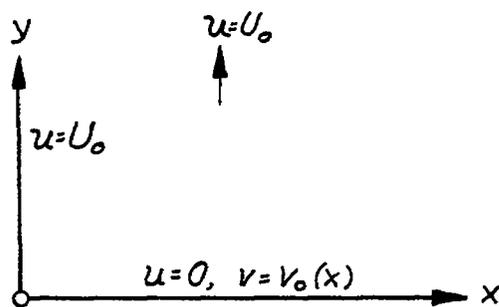


Figure 2.- Sketch of coordinates.

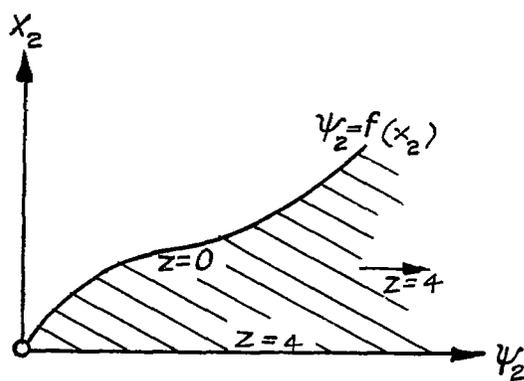


Figure 3.- Sketch of coordinates.

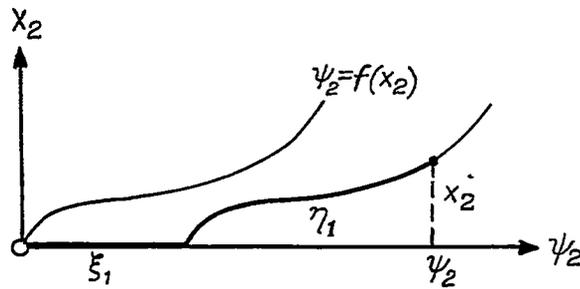


Figure 4.- Sketch of coordinates.

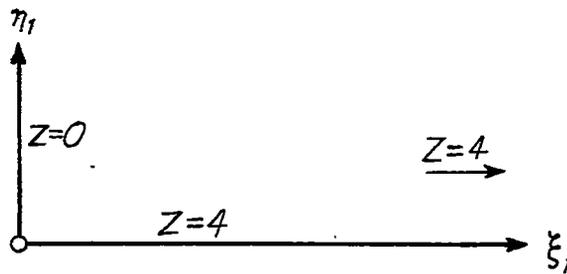


Figure 5.- Sketch of coordinates.



Figure 6.- Sketch of coordinates.

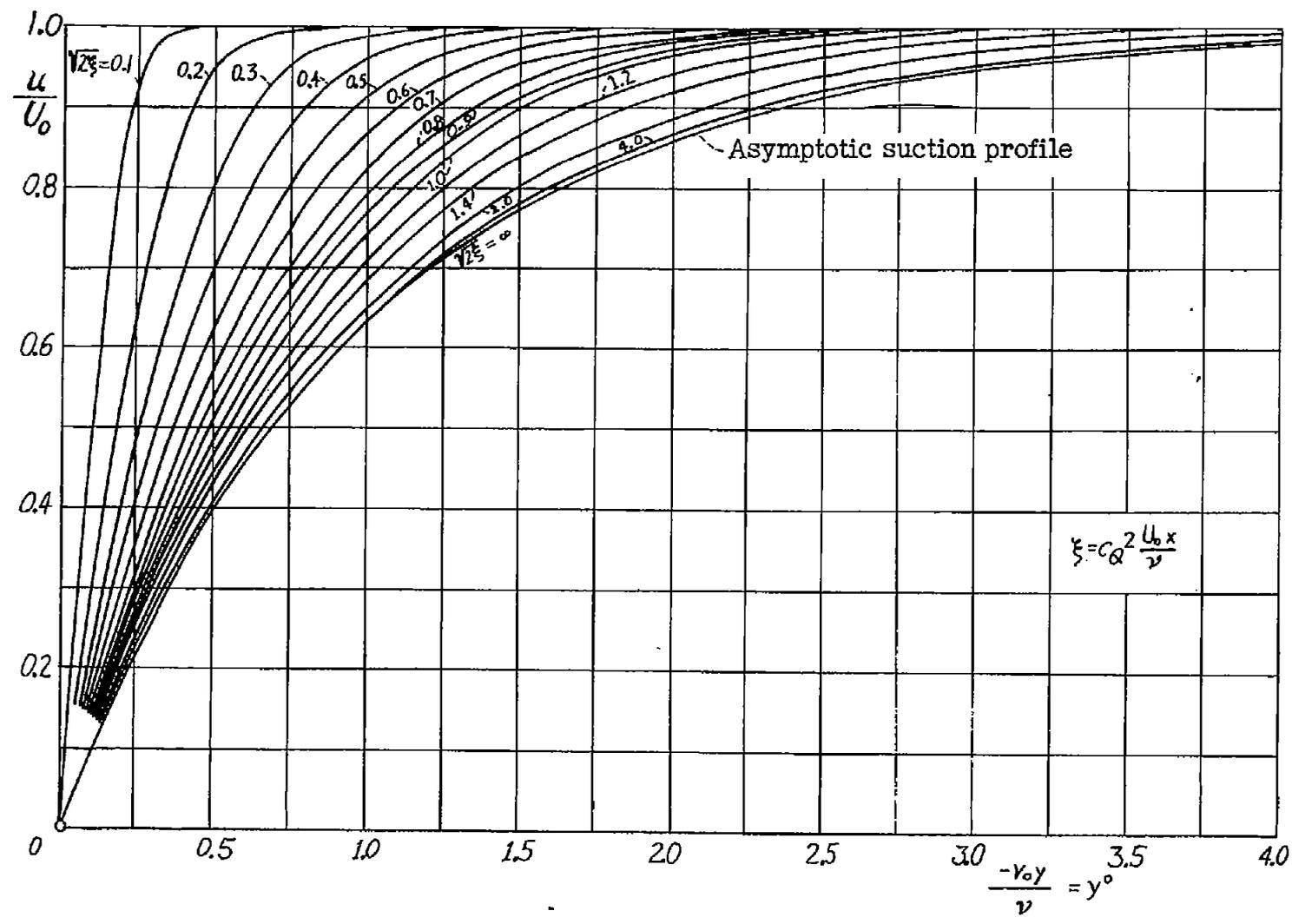


Figure 7.- Velocity distribution u/U_0 against $y^* = -v_0 y / \nu$.

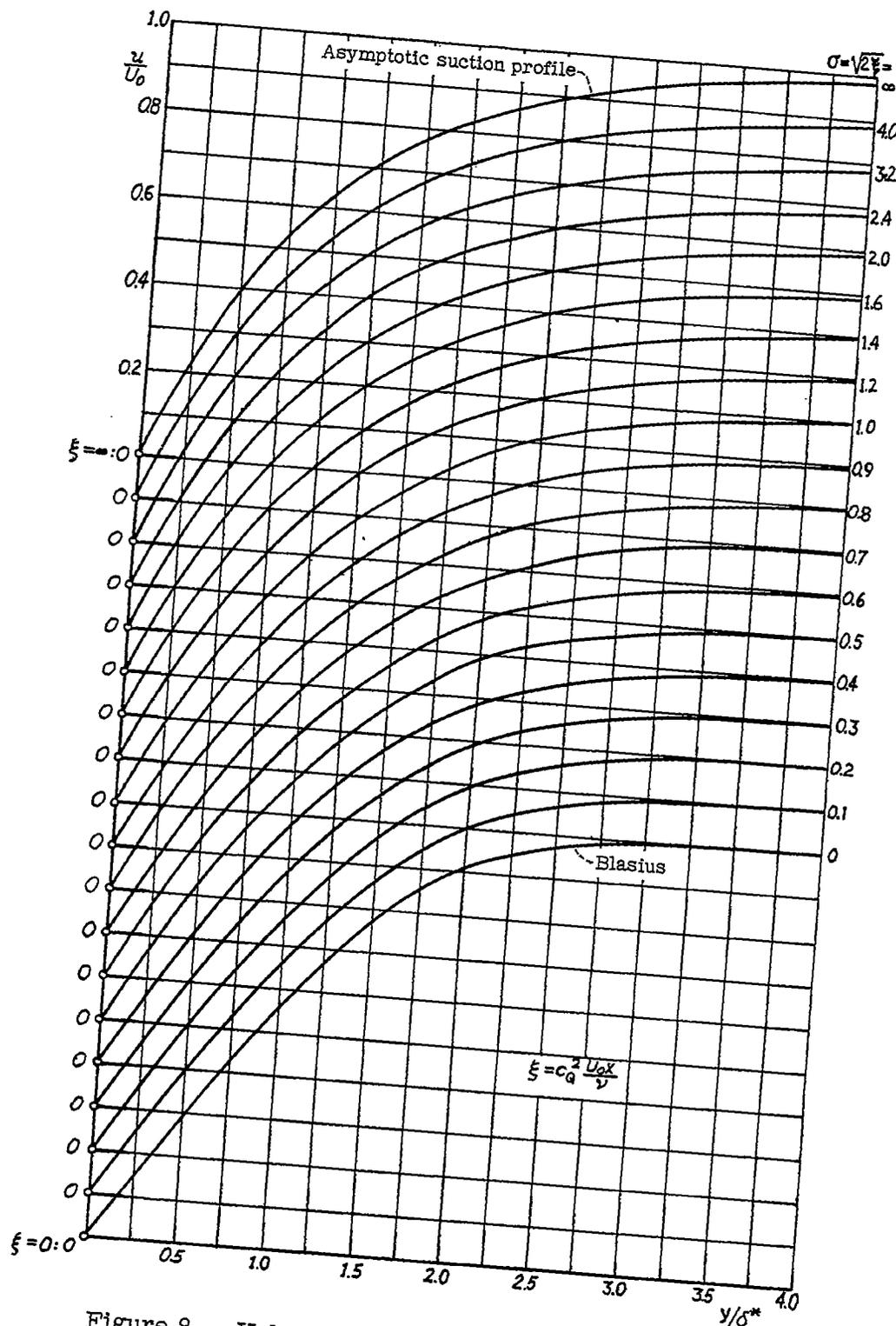


Figure 8.- Velocity distribution u/U_0 against y/δ^* .

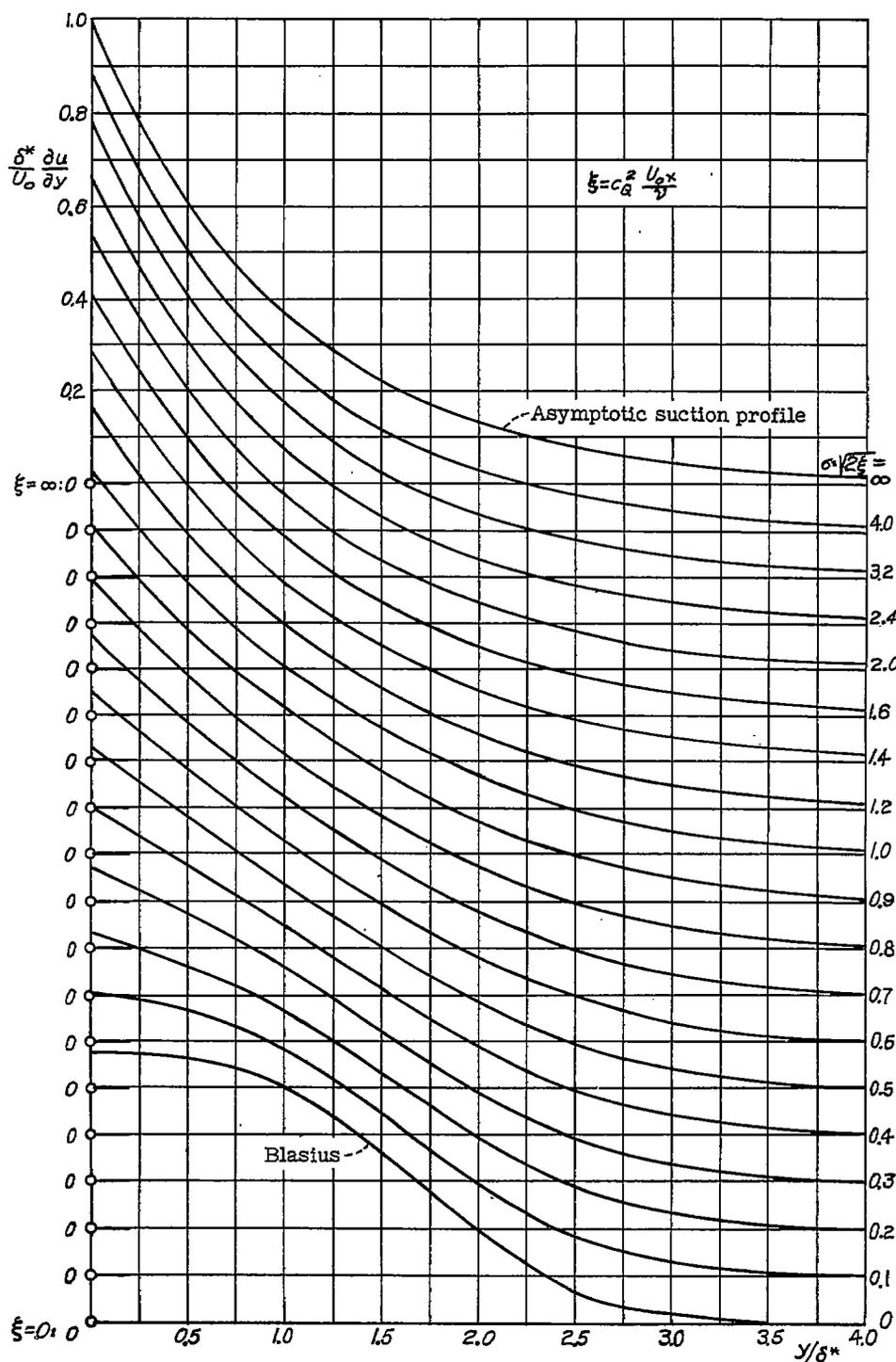


Figure 9.- The first derivative of the velocity distribution $\delta^*/U_0 \partial u/\partial y$ against y/δ^* .

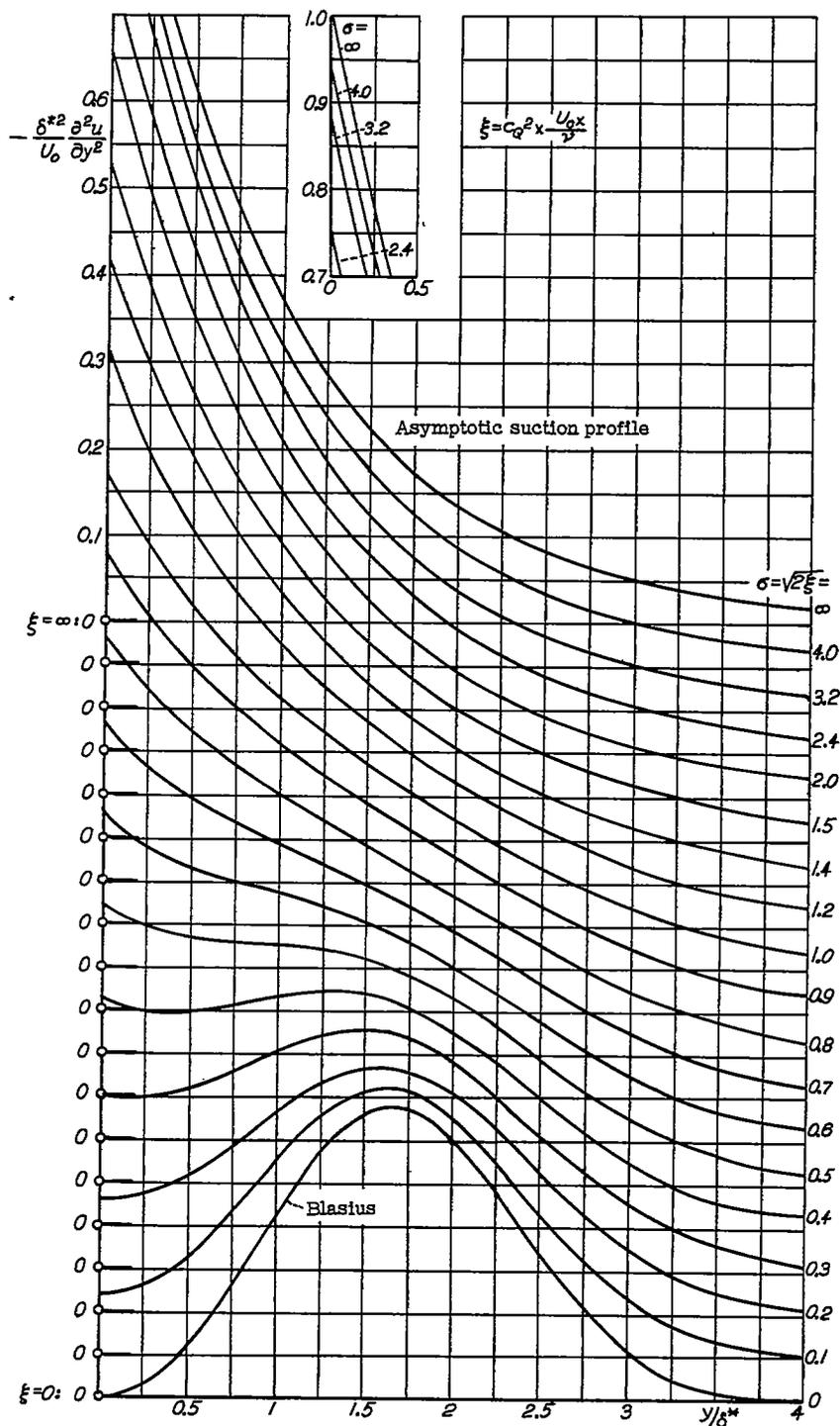


Figure 10.- The second derivative of the velocity distribution $\delta^{*2}/U_0 \partial^2 u/\partial y^2$ against y/δ^* .

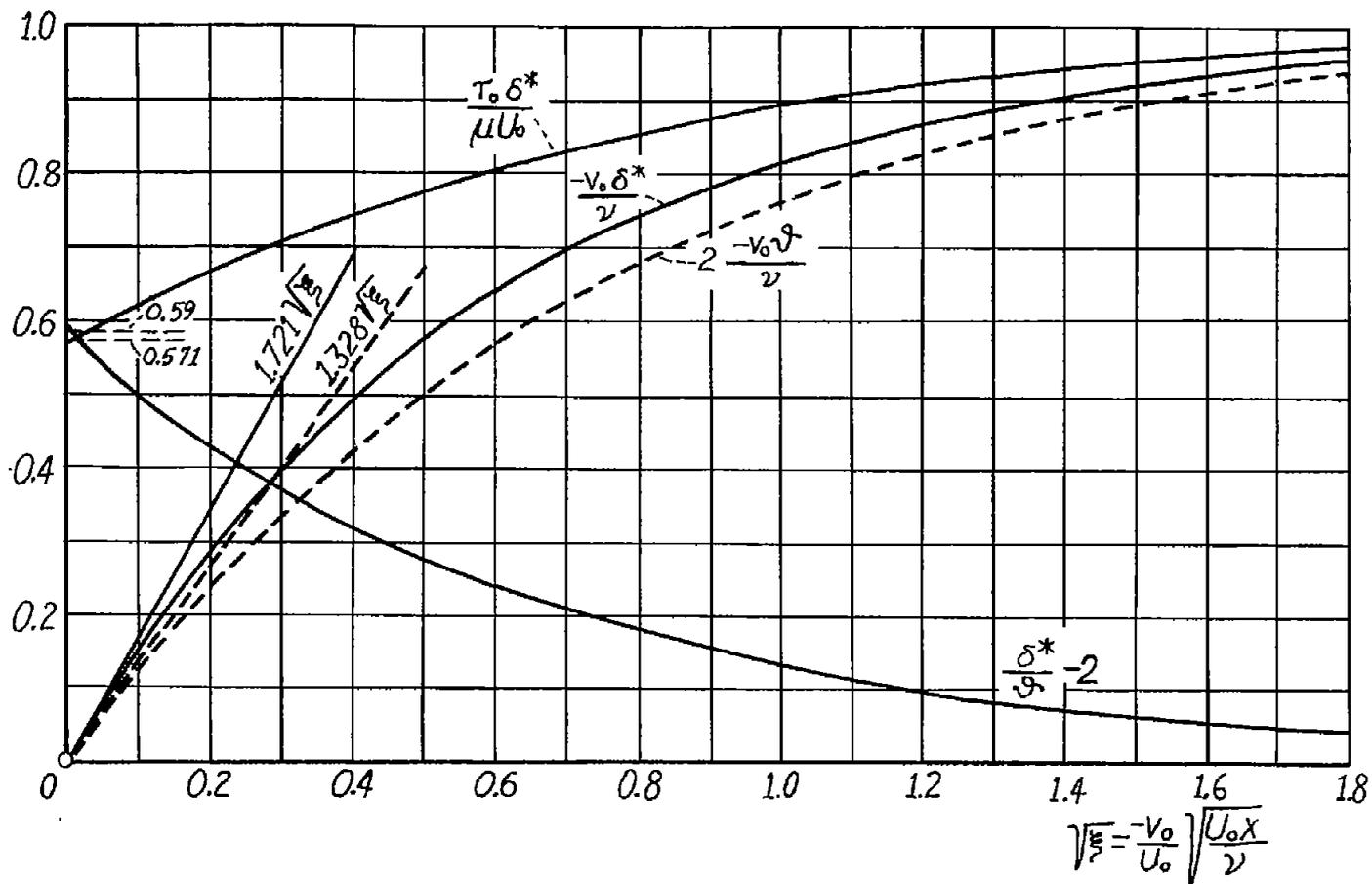


Figure 11.- The characteristic boundary-layer parameters $-\frac{v_0 \delta^*}{\nu}$, $-\frac{v_0 \delta}{\nu}$, $\frac{\delta^*}{\delta}$, $\frac{\tau_0 \delta^*}{\mu U_0}$ against $\sqrt{\xi}$.

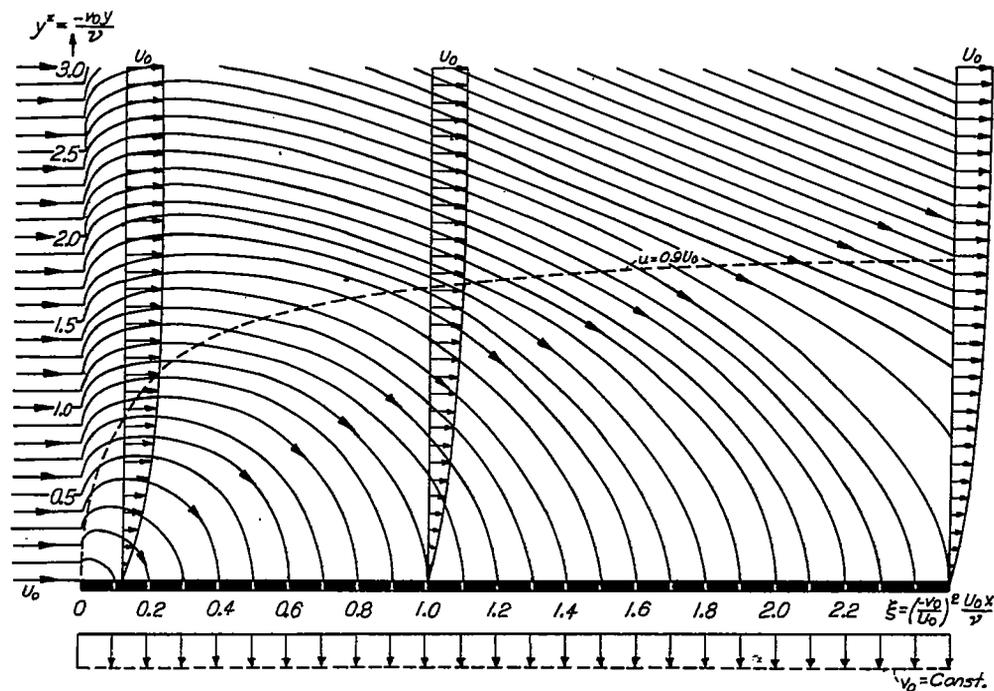


Figure 12.- Streamline pattern (total representation). The y-direction is, compared to the x-direction, increased by the factor $1/5 \ 1/c_Q$.

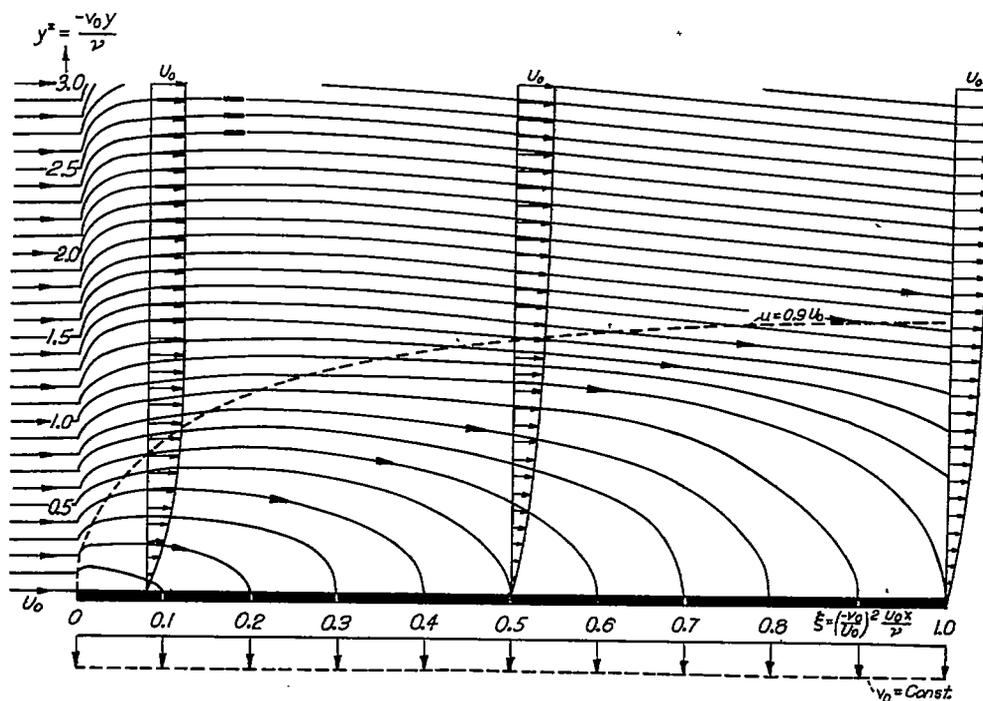


Figure 13.- Streamline pattern; representation at the leading edge of the plate magnified.

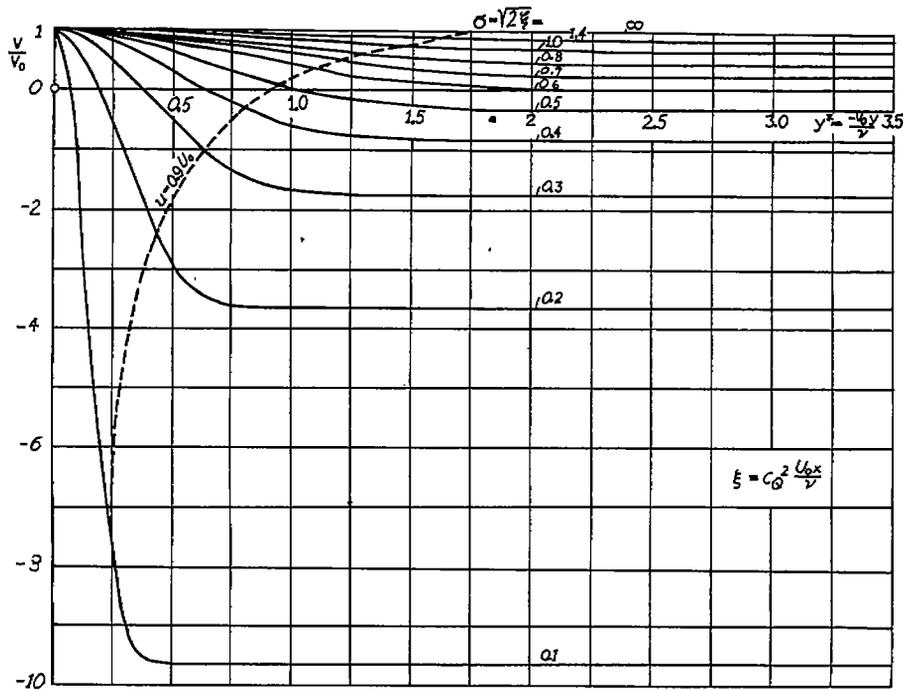


Figure 14.- The transverse velocity v/v_0 against y^* .

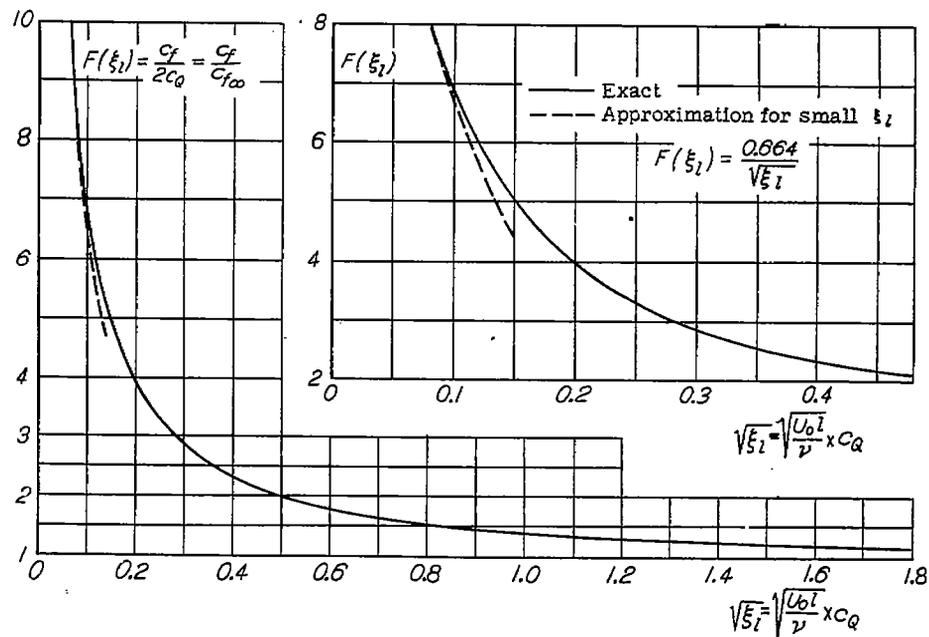


Figure 15.- The universal law for the friction drag of the plate with homogeneous suction $F(\xi_1) = c_f/2c_Q$ against $\xi_1 = c^2_Q U_0 l/\nu$.

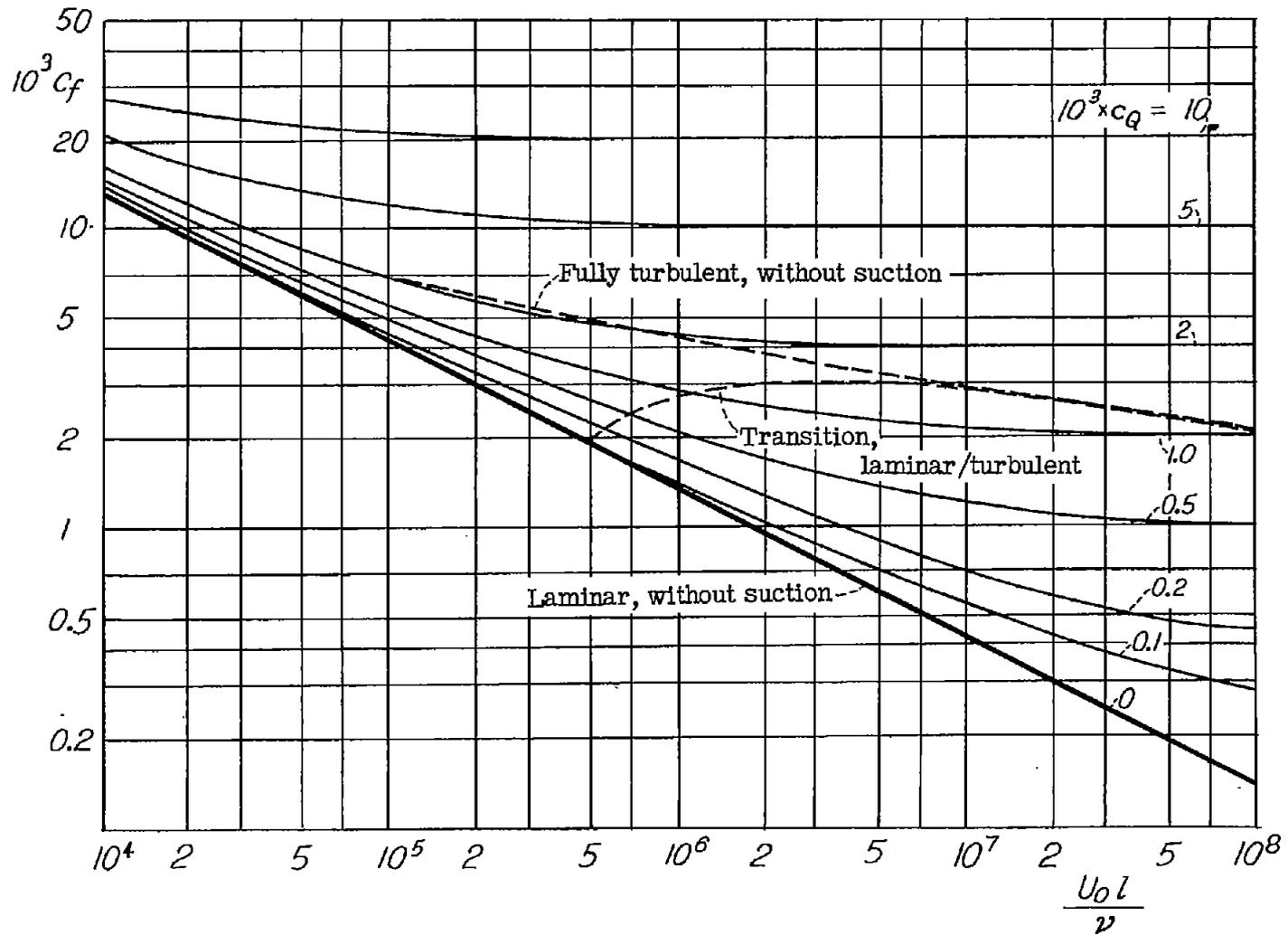


Figure 16.- The coefficient of the total friction drag c_f as a function of $Re = U_0 l / \nu$ for various mass coefficients c_Q .